ARCHIVE

for

RATIONAL MECHANICS

and

ANALYSIS

Edited by

C. TRUESDELL

Volume 4, Number 5

SPRINGER-VERLAG
BERLIN-GÖTTINGEN-HEIDELBERG
(Postverlagsort Berlin · 23. 3. 1960)

Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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The Mechanics of Non-Linear Materials with Memory

Part III

A. E. Green & R. S. Rivlin

1. Introduction

The mechanics of non-linear materials with memory has been discussed in previous papers [*1*, *2*] and an earlier report [*3*] ⋆. In these papers, it was assumed that, in a fixed coordinate system, the stress at a point of the material at any instant of time is determined by the deformation gradients at that instant and at previous instants. The limitations imposed on the form of the constitutive equation by the consideration that it is unaltered by a simultaneous rotation of the body and reference system are determined. In the present paper, we assume initially a constitutive equation in the form of implicit relations between the stress, its time derivatives and gradients of displacement, velocity, acceleration, *etc.* at a number of instants of time in some time interval. The limitations on the form of the constitutive equation resulting from the fact that it is unaltered by the imposition on the body of an additional arbitrary angular velocity, acceleration, *etc.* are discussed.

In §§ 6 and 7, we make the assumption that the stress in a fixed coordinate system at any instant of time depends explicitly on the gradients of the displacement, velocity, acceleration, *etc.* at that instant and previous instants. The limitations imposed on the form of such a constitutive equation by the consideration that it is unaltered by a simultaneous rotation of the body and the reference system are determined.

The remainder of the paper is concerned with the relation between such a constitutive equation and constitutive equations based on apparently more limited initial assumptions. It is shown in § 6 that if the dependence of the stress at any instant on the deformation history at preceding instants is sufficiently smooth, and if the motion starts from rest sufficiently smoothly, then there is no substantial loss of generality in assuming that the stress depends on the displacement gradients only at preceding instants and on the gradients of displacement, velocity, acceleration, *etc.* at the instant of measurement of the stress. In § 8, it is shown how various limitations in our initial assumption regarding the arguments on which the stress depends is reflected in limitations on the form of our resulting constitutive equation.

In § 9 it is shown how particular dependence of the stress on the gradients of displacement, velocity, acceleration *etc.* may be formally included in functional dependence of the stress on the displacement gradients only by the use of Dirac

⋆ The subject has also been discussed by Noll [*4*, *5*].

delta functions and their derivatives. Finally, in § 10, the relation of memory theories of the type discussed in the present paper to a previous theory of Rivlin & Ericksen [6] is discussed.

2. Kinematic Preliminaries

Let every particle of a three-dimensional body be at rest, prior to time $\tau=0$, relative to a fixed rectangular Cartesian coordinate system x. Let us suppose that after zero time the body is deformed continuously so that a typical particle P_0 of the body moves to P at time t and that at any time τ $(0\leqq\tau\leqq t)$ the coordinates of the particle in the system are $x_i(\tau)$. We shall employ the notation

$$X_i=x_i(0), \qquad x_i=x_i(t). \tag{2.1}$$

We consider that the functions $x_i(\tau)$ are specified as continuous functions of X_i and τ with continuous partial spatial derivatives up to any required order, except possibly at singular points, lines and surfaces. We then have

$$x_i(\tau)=x_i(X_j,\tau), \qquad x_i=x_i(X_j,t). \tag{2.2}$$

If this deformation is to be possible in a real material then

$$\left|\frac{\partial x_i(\tau)}{\partial X_j}\right|>0. \tag{2.3}$$

The components of velocity of the point P in our fixed Cartesian frame are denoted by $v_i^{(1)}(\tau)=v_i(\tau)$ and we use the notation $v_i(t)=v_i$. Thus

$$v_i^{(1)}(\tau)=v_i(\tau)=\frac{D x_i(\tau)}{D\tau} \tag{2.4}$$

where $D/D\tau$ denotes differentiation with respect to τ holding convected or material coordinates X_j fixed. The components of acceleration $v_i^{(2)}(\tau)$ are then given by

$$\begin{gathered} v_i^{(2)}(\tau)=\frac{D v_i^{(1)}(\tau)}{D\tau}=\frac{\partial v_i^{(1)}(\tau)}{\partial\tau}+v_m^{(1)}(\tau)\,\frac{\partial v_i^{(1)}(\tau)}{\partial x_m(\tau)}, \\ v_i^{(2)}=v_i^{(2)}(t), \end{gathered} \tag{2.5}$$

where $\partial/\partial\tau$ denotes differentiation with respect to time keeping $x_i(\tau)$ constant.

More generally, the $(n-1)^{\text{th}}$ acceleration components may be expressed as

$$\begin{gathered} v_i^{(n)}(\tau)=\frac{D v_i^{(n-1)}(\tau)}{D\tau}=\left[\frac{\partial}{\partial\tau}+v_m^{(1)}(\tau)\,\frac{\partial}{\partial x_m(\tau)}\right]v_i^{(n-1)}(\tau), \\ v_i^{(n)}=v_i^{(n)}(t), \qquad v_i^{(0)}(\tau)=x_i(\tau). \end{gathered} \tag{2.6}$$

We now consider another deformation of the body which differs from that described by (2.2) only by a superposed arbitrary rigid body rotation. Then the coordinates $\bar{x}_i(\tau)$ of the particle P_0 at time τ in the coordinate system x_i are given by

$$\bar{x}_i(\tau)=a_{ij}(\tau)\,x_j(\tau), \tag{2.7}$$

where $a_{ij}(\tau)$ are any set of continuous functions of τ defining a rigid body rotation and are therefore subject to the conditions

$$a_{ir}(\tau)\,a_{jr}(\tau)=a_{ri}(\tau)\,a_{rj}(\tau)=\delta_{ij}, \qquad |a_{ij}(\tau)|=1. \tag{2.8}$$

We denote the components of velocity, acceleration, ..., $(n-1)^{\text{th}}$ acceleration, in our fixed frame, due to the resultant of the deformations (2.2) and (2.7) by $\bar{v}_i^{(1)}(\tau) = \bar{v}_i(\tau)$, $\bar{v}_i^{(2)}(\tau), \ldots, \bar{v}_i^{(n)}(\tau)$ and use the notation $\bar{v}_i^{(s)}(t) = \bar{v}_i^{(s)}$ $(s = 1, 2, \ldots, n)$. Then

$$\bar{v}_i^{(s)}(\tau) = \frac{D\,\bar{v}_i^{(s-1)}(\tau)}{D\,\tau}, \qquad \bar{v}_i^{(0)}(\tau) = \bar{x}_i(\tau), \tag{2.9}$$

for $s = 1, 2, \ldots, n$.

From (2.6), (2.7) and (2.9) we have

$$\begin{aligned} \bar{v}_i(\tau) = \frac{D\,\bar{x}_i(\tau)}{D\,\tau} &= a_{ij}(\tau)\, v_j(\tau) + \frac{D\,a_{ij}(\tau)}{D\,\tau}\, x_j(\tau) \\ &= a_{ij}(\tau)\, v_j(\tau) + \frac{D\,a_{im}(\tau)}{D\,\tau}\, a_{jm}(\tau)\, \bar{x}_j(\tau). \end{aligned} \tag{2.10}$$

Also, from (2.8),

$$\frac{D\,a_{im}(\tau)}{D\,\tau}\, a_{jm}(\tau) + a_{im}(\tau)\, \frac{D\,a_{jm}(\tau)}{D\,\tau} = 0, \tag{2.11}$$

so that $\alpha_{ij}^{(1)}(\tau)$, defined by

$$\alpha_{ij}^{(1)}(\tau) = a_{jm}(\tau)\, \frac{D\,a_{im}(\tau)}{D\,\tau} = -\, a_{im}(\tau)\, \frac{D\,a_{jm}(\tau)}{D\,\tau}, \tag{2.12}$$

is skew symmetric and we may define a vector $\Omega_i^{(1)}(\tau)$ by the equations

$$\alpha_{ij}^{(1)}(\tau) = e_{jik}\, \Omega_k^{(1)}(\tau), \qquad 2\Omega_k^{(1)}(\tau) = e_{kij}\, \alpha_{ji}^{(1)}(\tau), \tag{2.13}$$

where $e_{ijk} = 1$ or -1, accordingly as i, j, k is an even or odd permutation of $1, 2, 3$ and $e_{ijk} = 0$ otherwise. With the help of (2.12) and (2.13) equation (2.10) becomes

$$\bar{v}_i(\tau) = a_{ij}(\tau)\, v_j(\tau) + e_{ikj}\, \Omega_k^{(1)}(\tau)\, \bar{x}_j(\tau) \tag{2.14}$$

and therefore $\Omega_i^{(1)}(\tau)$ is the component along the x_i axis of the angular velocity of the superposed rigid body motion at time τ. The angular acceleration, second angular acceleration, ... of the rigid body rotation at time τ are

$$\begin{aligned} \Omega_i^{(s)}(\tau) &= \frac{D^{s-1}\, \Omega_i^{(1)}(\tau)}{D\,\tau^{s-1}} \qquad (s = 2, 3, \ldots) \\ &= \tfrac{1}{2}\, e_{ijk}\, \alpha_{kj}^{(s)}(\tau), \end{aligned} \tag{2.15}$$

where

$$\alpha_{kj}^{(s)}(\tau) = \frac{D^{s-1}\, \alpha_{kj}^{(1)}(\tau)}{D\,\tau^{s-1}}. \tag{2.16}$$

If we substitute (2.12) into (2.16) we obtain

$$\alpha_{ij}^{(s)}(\tau) = \sum_{q=1}^{s-1} \binom{s-1}{q} \frac{D^q\, a_{jm}(\tau)}{D\,\tau^q}\, \frac{D^{s-q}\, a_{im}(\tau)}{D\,\tau^{s-q}}$$

and hence

$$\begin{aligned} a_{jm}(\tau)\, \frac{D^s\, a_{im}(\tau)}{D\,\tau^s} &= \alpha_{ij}^{(s)}(\tau) - \sum_{q=1}^{s-1} \binom{s-1}{q} \frac{D^q\, a_{jm}(\tau)}{D\,\tau^q}\, \frac{D^{s-q}\, a_{im}(\tau)}{D\,\tau^{s-q}} \\ &= \alpha_{ij}^{(s)}(\tau) - \sum_{q=1}^{s-1} \binom{s-1}{q} \left\{ a_{km}(\tau)\, \frac{D^q\, a_{jm}(\tau)}{D\,\tau^q} \right\} \left\{ a_{kn}(\tau)\, \frac{D^{s-q}\, a_{in}(\tau)}{D\,\tau^{s-q}} \right\}. \end{aligned} \tag{2.17}$$

By successive application of (2.17) we see that

$$a_{jm}(\tau)\, \frac{D^s a_{im}(\tau)}{D\,\tau^s} \tag{2.18}$$

may be expressed as the sum of $\alpha_{ij}^{(s)}(\tau)$ and a polynomial in $\alpha_{ij}^{(1)}(\tau), \alpha_{ij}^{(2)}(\tau), \ldots, \alpha_{ij}^{(s-1)}(\tau)$, and we recall from (2.15) that the $(s-1)^{\text{th}}$ angular acceleration of the rigid body motion at time τ can be expressed in terms of the three non-zero components of $\alpha_{ij}^{(s)}(\tau)$.

3. Tensors Dependent on Rotation

From (2.10) we have

$$\begin{aligned}\frac{\partial \bar{v}_i(\tau)}{\partial \bar{x}_j(\tau)} &= a_{ir}(\tau)\, a_{js}(\tau)\, \frac{\partial v_r(\tau)}{\partial x_s(\tau)} + a_{jm}(\tau)\, \frac{D a_{im}(\tau)}{D\tau} \\ &= a_{ir}(\tau)\, a_{js}(\tau)\, \frac{\partial v_r(\tau)}{\partial x_s(\tau)} - a_{im}(\tau)\, \frac{D a_{jm}(\tau)}{D\tau},\end{aligned} \tag{3.1}$$

so that

$$\begin{aligned}\frac{D a_{ij}(\tau)}{D\tau} &= a_{mj}(\tau)\, \frac{\partial \bar{v}_i(\tau)}{\partial \bar{x}_m(\tau)} - a_{im}(\tau)\, \frac{\partial v_m(\tau)}{\partial x_j(\tau)} \\ &= -a_{mj}(\tau)\, \frac{\partial \bar{v}_m(\tau)}{\partial \bar{x}_i(\tau)} + a_{im}(\tau)\, \frac{\partial v_j(\tau)}{\partial x_m(\tau)}.\end{aligned} \tag{3.2}$$

Using the notation

$$\begin{aligned}\omega_{ij}(\tau) = \omega_{ij}^{(1)}(\tau) &= \frac{1}{2}\left\{\frac{\partial v_i(\tau)}{\partial x_j(\tau)} - \frac{\partial v_j(\tau)}{\partial x_i(\tau)}\right\}, \\ \bar{\omega}_{ij}(\tau) = \bar{\omega}_{ij}^{(1)}(\tau) &= \frac{1}{2}\left\{\frac{\partial \bar{v}_i(\tau)}{\partial \bar{x}_j(\tau)} - \frac{\partial \bar{v}_j(\tau)}{\partial \bar{x}_i(\tau)}\right\},\end{aligned} \tag{3.3}$$

we see, with the help of (3.1) and (2.12), that

$$\bar{\omega}_{ij}(\tau) = a_{ir}(\tau)\, a_{js}(\tau)\, \omega_{rs}(\tau) + \alpha_{ij}^{(1)}(\tau). \tag{3.4}$$

If, at a particular time $\tau = \tau_\alpha$, the body is rotating but occupies the same instantaneous position at this time as that given by (2.2), we have $a_{ij}(\tau_\alpha) = \delta_{ij}$ and

$$\bar{\omega}_{ij}(\tau_\alpha) = \omega_{ij}(\tau_\alpha) + \alpha_{ij}^{(1)}(\tau_\alpha). \tag{3.5}$$

Hence $\omega_{ij}(\tau_\alpha)$ is not independent of the angular velocity of a superposed rigid body rotation at time τ_α.

More generally suppose

$$\begin{aligned}\omega_{ij}^{(s)}(\tau) &= \frac{1}{2}\left\{\frac{\partial v_i^{(s)}(\tau)}{\partial x_j(\tau)} - \frac{\partial v_j^{(s)}(\tau)}{\partial x_i(\tau)}\right\}, \\ \bar{\omega}_{ij}^{(s)}(\tau) &= \frac{1}{2}\left\{\frac{\partial \bar{v}_i^{(s)}(\tau)}{\partial \bar{x}_j(\tau)} - \frac{\partial \bar{v}_i^{(s)}(\tau)}{\partial \bar{x}_i(\tau)}\right\},\end{aligned} \tag{3.6}$$

for $s = 1, 2, \ldots$. From (2.6) to (2.9) we have

$$\begin{aligned}\bar{v}_i^{(s)}(\tau) &= \sum_{q=0}^{s} \binom{s}{q} \frac{D^{s-q} a_{im}(\tau)}{D\tau^{s-q}}\, v_m^{(q)}(\tau), \\ \frac{\partial \bar{v}_i^{(s)}(\tau)}{\partial \bar{x}_j(\tau)} &= \sum_{q=0}^{s} \binom{s}{q} a_{jk}(\tau)\, \frac{D^{s-q} a_{im}(\tau)}{D\tau^{s-q}}\, \frac{\partial v_m^{(q)}(\tau)}{\partial x_k(\tau)},\end{aligned} \tag{3.7}$$

where $v_i^{(0)}(\tau) = x_i(\tau)$, $\bar{v}_i^{(0)}(\tau) = \bar{x}_i(\tau)$, and therefore

$$\begin{aligned}\bar{\omega}_{ij}^{(s)}(\tau) = a_{im}(\tau)\, a_{jn}(\tau)\, \omega_{mn}^{(s)}(\tau) + \frac{1}{2}\left\{a_{jk}(\tau)\, \frac{D^s a_{ik}(\tau)}{D\tau^s} - a_{ik}(\tau)\, \frac{D^s a_{jk}(\tau)}{D\tau^s}\right\} + \\ + \frac{1}{2}\sum_{q=1}^{s-1} \binom{s}{q}\left\{a_{jk}(\tau)\, \frac{D^{s-q} a_{im}(\tau)}{D\tau^{s-q}}\, \frac{\partial v_m^{(q)}(\tau)}{\partial x_k(\tau)} - a_{ik}(\tau)\, \frac{D^{s-q} a_{jm}(\tau)}{D\tau^{s-q}}\, \frac{\partial v_m^{(q)}(\tau)}{\partial x_k(\tau)}\right\}.\end{aligned} \tag{3.8}$$

If, at a particular time $\tau=\tau_\alpha$, the body is rotating but occupies the same instantaneous position at this time as that given by (2.2), we have $a_{ij}(\tau_\alpha)=\delta_{ij}$ and

$$\begin{aligned}\bar{\omega}_{ij}^{(s)}(\tau_\alpha) = \omega_{ij}^{(s)}(\tau_\alpha) + \frac{1}{2}\left\{\frac{D^s a_{ij}(\tau_\alpha)}{D\tau_\alpha^s} - \frac{D^s a_{ji}(\tau_\alpha)}{D\tau_\alpha^s}\right\} + \\ + \frac{1}{2}\sum_{q=1}^{s-1}\binom{s}{q}\left\{\frac{D^{s-q} a_{im}(\tau_\alpha)}{D\tau_\alpha^{s-q}}\frac{\partial v_m^{(q)}(\tau_\alpha)}{\partial x_j(\tau_\alpha)} - \frac{D^{s-q} a_{jm}(\tau_\alpha)}{D\tau_\alpha^{s-q}}\frac{\partial v_m^{(q)}(\tau_\alpha)}{\partial x_i(\tau_\alpha)}\right\}.\end{aligned} \tag{3.9}$$

Also, from (2.17) and (2.18) with $a_{ij}(\tau_\alpha)=\delta_{ij}$, the quantities $D^s a_{ij}(\tau_\alpha)/D\tau_\alpha^s$ $(s=1, 2, \ldots)$ can be expressed as the sum of $\alpha_{ij}^{(s)}(\tau_\alpha)$ and a polynomial in $\alpha_{ij}^{(1)}(\tau_\alpha)$, $\ldots, \alpha_{ij}^{(s-1)}(\tau_\alpha)$. Hence $\bar{\omega}_{ij}^{(s)}(\tau_\alpha)$ is not equal to $\omega_{ij}^{(s)}(\tau_\alpha)$ but depends also on the three components of the $(s-1)^{\text{th}}$ angular acceleration of the superposed rigid body motion at time τ_α, and on lower order angular accelerations.

4. Tensors Independent of Rotation

We denote the components of stress in the coordinate system x_i at the particle P_0 at time τ by $\sigma_{ij}(\tau)$ and employ the notation

$$\sigma_{ij}=\sigma_{ij}(t). \tag{4.1}$$

Now consider a deformation given by (2.2) and an arbitrary superposed rigid body rotation (2.7). The stress components at time τ in the coordinate system x_i due to the resultant of the deformations described by (2.2) and (2.7) are denoted by $\bar{\sigma}_{ij}(\tau)$. The stress components at the particle P_0 at time τ, referred to a moving rectangular Cartesian coordinate system which coincides with the fixed system x_i at time $\tau=0$ and rotates with the rigid body motion described by (2.7), are still $\sigma_{ij}(\tau)$. Hence, in view of the tensor character of stress components

$$\bar{\sigma}_{ij}(\tau) = a_{ir}(\tau)\, a_{js}(\tau)\, \sigma_{rs}(\tau). \tag{4.2}$$

From (4.2) we have

$$\frac{D\bar{\sigma}_{ij}(\tau)}{D\tau} = a_{ir}(\tau)\, a_{js}(\tau)\frac{D\sigma_{rs}(\tau)}{D\tau} + a_{ir}(\tau)\frac{D a_{js}(\tau)}{D\tau}\sigma_{rs}(\tau) + \frac{D a_{ir}(\tau)}{D\tau} a_{js}(\tau)\, \sigma_{rs}(\tau).$$

With the help of (3.2) this equation yields

$$\begin{aligned}\frac{D\bar{\sigma}_{ij}(\tau)}{D\tau} - \bar{\sigma}_{im}(\tau)\frac{\partial \bar{v}_j(\tau)}{\partial \bar{x}_m(\tau)} - \bar{\sigma}_{mj}(\tau)\frac{\partial \bar{v}_i(\tau)}{\partial \bar{x}_m(\tau)} \\ = a_{ir}(\tau)\, a_{js}(\tau)\left[\frac{D\sigma_{rs}(\tau)}{D\tau} - \sigma_{rm}(\tau)\frac{\partial v_s(\tau)}{\partial x_m(\tau)} - \sigma_{ms}(\tau)\frac{\partial v_r(\tau)}{\partial x_m(\tau)}\right],\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{D\bar{\sigma}_{ij}(\tau)}{D\tau} + \bar{\sigma}_{im}(\tau)\frac{\partial \bar{v}_m(\tau)}{\partial \bar{x}_j(\tau)} + \bar{\sigma}_{mj}(\tau)\frac{\partial \bar{v}_m(\tau)}{\partial \bar{x}_i(\tau)} \\ = a_{ir}(\tau)\, a_{js}(\tau)\left[\frac{D\sigma_{rs}(\tau)}{D\tau} + \sigma_{rm}(\tau)\frac{\partial v_m(\tau)}{\partial x_s(\tau)} + \sigma_{ms}(\tau)\frac{\partial v_m(\tau)}{\partial x_r(\tau)}\right].\end{aligned} \tag{4.4}$$

Also, by adding (4.3) and (4.4), and using (3.3), we have

$$\begin{aligned}\frac{D\bar{\sigma}_{ij}(\tau)}{D\tau} + \bar{\sigma}_{im}(\tau)\,\bar{\omega}_{mj}(\tau) + \bar{\sigma}_{mj}(\tau)\,\bar{\omega}_{mi}(\tau) \\ = a_{ir}(\tau)\, a_{js}(\tau)\left[\frac{D\sigma_{rs}(\tau)}{D\tau} + \sigma_{rm}(\tau)\,\omega_{ms}(\tau) + \sigma_{ms}(\tau)\,\omega_{mr}(\tau)\right].\end{aligned} \tag{4.5}$$

If we now introduce the notation

$$\sigma_{ij}^{(1)}(\tau) = \frac{D\,\sigma_{ij}(\tau)}{D\tau} + \sigma_{im}(\tau)\,\frac{\partial v_m(\tau)}{\partial x_j(\tau)} + \sigma_{mj}(\tau)\,\frac{\partial v_m(\tau)}{\partial x_i(\tau)},$$
$$\sigma_{ij}^{*(1)}(\tau) = \frac{D\,\sigma_{ij}(\tau)}{D\tau} - \sigma_{im}(\tau)\,\frac{\partial v_j(\tau)}{\partial x_m(\tau)} - \sigma_{mj}(\tau)\,\frac{\partial v_i(\tau)}{\partial x_m(\tau)}, \tag{4.6}$$
$$\frac{\mathscr{D}\,\sigma_{ij}(\tau)}{\mathscr{D}\tau} = \frac{D\,\sigma_{ij}(\tau)}{D\tau} + \sigma_{im}(\tau)\,\omega_{mj}(\tau) + \sigma_{mj}(\tau)\,\omega_{mi}(\tau),$$

it follows from (4.3) to (4.5) that

$$\bar{\sigma}_{ij}^{(1)}(\tau) = a_{im}(\tau)\,a_{jn}(\tau)\,\sigma_{mn}^{(1)}(\tau),$$
$$\bar{\sigma}_{ij}^{*(1)}(\tau) = a_{im}(\tau)\,a_{jn}(\tau)\,\sigma_{mn}^{*(1)}(\tau), \tag{4.7}$$
$$\frac{\mathscr{D}\bar{\sigma}_{ij}(\tau)}{\mathscr{D}\tau} = a_{im}(\tau)\,a_{jn}(\tau)\,\frac{\mathscr{D}\,\sigma_{mn}(\tau)}{\mathscr{D}\tau}.$$

The results (4.6) and (4.7) have been derived directly from (4.2) using also (3.3) and are valid whether or not $\sigma_{ij}(\tau)$ is the stress tensor, provided it satisfies an equation of the form (4.2). Hence by repeated differentiation using (4.7), we find that

$$\bar{\sigma}_{ij}^{(s)}(\tau) = a_{im}(\tau)\,a_{jn}(\tau)\,\sigma_{mn}^{(s)}(\tau),$$
$$\bar{\sigma}_{ij}^{*(s)}(\tau) = a_{im}(\tau)\,a_{jn}(\tau)\,\sigma_{mn}^{*(s)}(\tau), \tag{4.8}$$
$$\frac{\mathscr{D}^s\bar{\sigma}_{ij}(\tau)}{\mathscr{D}\tau^s} = a_{im}(\tau)\,a_{jn}(\tau)\,\frac{\mathscr{D}^s\,\sigma_{mn}(\tau)}{\mathscr{D}\tau^s},$$

for $s = 1, 2, \ldots$, where

$$\sigma_{ij}^{(s+1)}(\tau) = \frac{D\,\sigma_{ij}^{(s)}(\tau)}{D\tau} + \sigma_{im}^{(s)}(\tau)\,\frac{\partial v_m(\tau)}{\partial x_j(\tau)} + \sigma_{mj}^{(s)}(\tau)\,\frac{\partial v_m(\tau)}{\partial x_i(\tau)},$$
$$\sigma_{ij}^{*(s+1)}(\tau) = \frac{D\,\sigma_{ij}^{*(s)}(\tau)}{D\tau} - \sigma_{im}^{*(s)}(\tau)\,\frac{\partial v_j(\tau)}{\partial x_m(\tau)} - \sigma_{mj}^{*(s)}(\tau)\,\frac{\partial v_i(\tau)}{\partial x_m(\tau)}, \tag{4.9}$$
$$\frac{\mathscr{D}^{s+1}\sigma_{ij}(\tau)}{\mathscr{D}\tau^{s+1}} = \frac{D}{D\tau}\left\{\frac{\mathscr{D}^s\,\sigma_{ij}(\tau)}{\mathscr{D}\tau^s}\right\} + \frac{\mathscr{D}^s\,\sigma_{im}(\tau)}{\mathscr{D}\tau^s}\,\omega_{mj}(\tau) + \frac{\mathscr{D}^s\,\sigma_{mj}(\tau)}{\mathscr{D}\tau^s}\,\omega_{mi}(\tau)$$

and

$$\sigma_{ij}^{(0)}(\tau) = \sigma_{ij}^{*(0)}(\tau) = \sigma_{ij}(\tau).$$

We now obtain an alternative expression for $\sigma_{ij}^{(s)}(\tau)$ which will be of use later. From (2.7), (2.8) and (4.2) we see that

$$\bar{\sigma}_{ij}(\tau)\,d\bar{x}_i(\tau)\,d\bar{x}_j(\tau) = \sigma_{ij}(\tau)\,d\,x_i(\tau)\,d\,x_j(\tau)$$

so that $\beta(\tau)$, defined by

$$\beta(\tau) = \sigma_{ij}(\tau)\,d\,x_i(\tau)\,d\,x_j(\tau) = t_{mn}(\tau)\,d\,X_m\,d\,X_n, \tag{4.10}$$

is independent of rigid body rotations superposed on the original motion, where $t_{mn}(\tau)$, defined by

$$t_{mn}(\tau) = \sigma_{ij}(\tau)\,\frac{\partial x_i(\tau)}{\partial X_m}\,\frac{\partial x_j(\tau)}{\partial X_n}, \tag{4.11}$$

are functions independent of rigid body rotations, for all values of τ. This is also evident from the fact that $t_{mn}(\tau)$ are the covariant components of the stress tensor referred to curvilinear coordinates X_m in the deformed body at time τ.

From (4.10) we have

$$\frac{D\beta(\tau)}{D\tau}=\frac{D\sigma_{ij}(\tau)}{D\tau}dx_i(\tau)\,dx_j(\tau)+\sigma_{ij}(\tau)\frac{D\{dx_i(\tau)\}}{D\tau}dx_j(\tau)+\sigma_{ij}(\tau)\,dx_i(\tau)\frac{D\{dx_j(\tau)\}}{D\tau}$$
$$=\frac{Dt_{mn}(\tau)}{D\tau}dX_m\,dX_n.$$

Since

$$\frac{D}{D\tau}dx_i(\tau)=dv_i(\tau)=\frac{\partial v_i(\tau)}{\partial x_j(\tau)}dx_j(\tau)$$

we have, using (4.6),

$$\frac{D\beta(\tau)}{D\tau}=\sigma_{ij}^{(1)}(\tau)\,dx_i(\tau)\,dx_j(\tau)=\frac{Dt_{mn}(\tau)}{D\tau}dX_m\,dX_n. \tag{4.12}$$

Repeated differentiation of (4.12) gives

$$\frac{D^s\beta(\tau)}{D\tau^s}=\sigma_{ij}^{(s)}(\tau)\,dx_i(\tau)\,dx_j(\tau)=\frac{D^s t_{mn}(\tau)}{D\tau^s}dX_m\,dX_n. \tag{4.13}$$

Returning to (4.10) we see that

$$\frac{D^s\beta(\tau)}{D\tau^s}=\sum_{q=0}^{s}\left[\binom{s}{q}\frac{D^{s-q}\sigma_{ij}(\tau)}{D\tau^{s-q}}\sum_{k=0}^{q}\binom{q}{k}\frac{D^{q-k}dx_i(\tau)}{D\tau^{q-k}}\,\frac{D^k dx_j(\tau)}{D\tau^k}\right]. \tag{4.14}$$

Since

$$\frac{D^k dx_i(\tau)}{D\tau^k}=dv_i^{(k)}(\tau)=\frac{\partial v_i^{(k)}(\tau)}{\partial x_g(\tau)}dx_g(\tau)$$

and $\partial v_i^{(0)}(\tau)/\partial x_j(\tau)=\delta_{ij}$, equations (4.14) and (4.13) yield the alternative formula

$$\begin{aligned}\sigma_{ij}^{(s)}(\tau)&=\sum_{q=0}^{s}\left[\binom{s}{q}\frac{D^{s-q}\sigma_{mn}(\tau)}{D\tau^{s-q}}\sum_{k=0}^{q}\binom{q}{k}\frac{\partial v_m^{(q-k)}(\tau)}{\partial x_i(\tau)}\,\frac{\partial v_n^{(k)}(\tau)}{\partial x_j(\tau)}\right]\\&=\frac{D^s\sigma_{ij}(\tau)}{D\tau^s}+\sum_{q=1}^{s}\left[\binom{s}{q}\frac{D^{s-q}\sigma_{mn}(\tau)}{D\tau^{s-q}}\sum_{k=0}^{q}\binom{q}{k}\frac{\partial v_m^{(q-k)}(\tau)}{\partial x_i(\tau)}\,\frac{\partial v_n^{(k)}(\tau)}{\partial x_j(\tau)}\right]\end{aligned} \tag{4.15}$$

for $\sigma_{ij}^{(s)}(\tau)$. Also, from (4.13),

$$\frac{D^s t_{mn}(\tau)}{D\tau^s}=\sigma_{ij}^{(s)}(\tau)\frac{\partial x_i(\tau)}{\partial X_m}\,\frac{\partial x_j(\tau)}{\partial X_n} \tag{4.16}$$

and these functions are completely independent of any superposed rigid body rotations.

Again, if

$$t^{mn}(\tau)=\bar{\sigma}_{ij}(\tau)\frac{\partial X_m}{\partial\bar{x}_i(\tau)}\,\frac{\partial X_n}{\partial\bar{x}_j(\tau)}=\sigma_{ij}(\tau)\frac{\partial X_m}{\partial x_i(\tau)}\,\frac{\partial X_n}{\partial x_j(\tau)} \tag{4.17}$$

then $t^{mn}(\tau)$ are independent of rigid body rotations for all time τ and we may show that

$$\frac{D^s t^{mn}(\tau)}{D\tau^s}=\sigma_{ij}^{*(s)}(\tau)\frac{\partial X_m}{\partial x_i(\tau)}\,\frac{\partial X_n}{\partial x_j(\tau)}. \tag{4.18}$$

The functions $t^{mn}(\tau)$ are contravariant components of the stress tensor at time τ referred to curvilinear coordinates X_m in the deformed body at this time.

When $\sigma_{ij}(\tau)$, instead of denoting the stress tensor, has the special value δ_{ij} we denote the corresponding values of $\sigma_{ij}^{(s)}(\tau)$ and $t_{mn}(\tau)$ by $A_{ij}^{(s)}(\tau)$ and $g_{mn}(\tau)$

respectively. Hence, from (4.6), (4.8), (4.9), (4.15) and (4.16), we have

$$\begin{aligned}
2A_{ij}(\tau) &= A_{ij}^{(1)}(\tau) = \frac{\partial v_i(\tau)}{\partial x_j(\tau)} + \frac{\partial v_j(\tau)}{\partial x_i(\tau)}, \qquad A_{ij}^{(0)}(\tau) = \delta_{ij}, \\
A_{ij}^{(s)}(\tau) &= \frac{D A_{ij}^{(s-1)}(\tau)}{D\tau} + A_{im}^{(s-1)}(\tau)\frac{\partial v_m(\tau)}{\partial x_j(\tau)} + A_{mj}^{(s-1)}(\tau)\frac{\partial v_m(\tau)}{\partial x_i(\tau)}, \\
&= \frac{\partial v_i^{(s)}(\tau)}{\partial x_j(\tau)} + \frac{\partial v_j^{(s)}(\tau)}{\partial x_i(\tau)} + \sum_{q=1}^{s-1}\binom{s}{q}\frac{\partial v_m^{(s-q)}(\tau)}{\partial x_i(\tau)}\frac{\partial v_m^{(q)}(\tau)}{\partial x_j(\tau)},
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\bar{A}_{ij}^{(s)}(\tau) &= a_{im}(\tau)\, a_{jn}(\tau)\, A_{mn}^{(s)}(\tau), \\
\frac{D^s g_{mn}(\tau)}{D\tau^s} &= A_{ij}^{(s)}(\tau)\frac{\partial x_i(\tau)}{\partial X_m}\frac{\partial x_j(\tau)}{\partial X_n}, \\
g_{mn}(\tau) &= \frac{\partial x_i(\tau)}{\partial X_m}\frac{\partial x_i(\tau)}{\partial X_n}.
\end{aligned} \tag{4.20}$$

When $\sigma_{ij}(\tau) = \delta_{ij}$ we observe from (4.6) that

$$\frac{\mathscr{D}\delta_{ij}}{\mathscr{D}\tau} = 0. \tag{4.21}$$

This relation is of importance when we express our results in terms of a general fixed curvilinear system of coordinates.

If at time $\tau = \tau_\alpha$ the body has a superposed rigid body rotation but occupies the same instantaneous position as that given by (4.2) (with $\tau = \tau_\alpha$) we see from (4.8) and (4.19) that

$$\bar{\sigma}_{ij}^{(s)}(\tau_\alpha) = \sigma_{ij}^{(s)}(\tau_\alpha), \qquad \bar{A}_{ij}^{(s)}(\tau_\alpha) = A_{ij}^{(s)}(\tau_\alpha) \qquad (s = 0, 1, 2, \ldots). \tag{4.22}$$

5. General Constitutive Equations

We consider general constitutive relations of the form*

$$\begin{aligned}
f_{ij}\Bigg[&\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, \frac{\partial v_p^{(1)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}, \ldots, \frac{\partial v_p^{(n)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}; \\
&\sigma_{pq}(\tau_\alpha), \frac{D\sigma_{pq}(\tau_\alpha)}{D\tau_\alpha}, \ldots, \frac{D^m\sigma_{pq}(\tau_\alpha)}{D\tau_\alpha^m}\Bigg] = 0,
\end{aligned} \tag{5.1}$$

where τ_α ($\alpha = 0, 1, 2, \ldots, N$) denote $N+1$ distinct instants of time between $\tau = 0$ and $\tau = t$, and where $\tau_0 = t$. Also we assume $f_{ji} = f_{ij}$. From (4.15) we see that $D^m\sigma_{pq}(\tau_\alpha)/D\tau_\alpha^m$ can be expressed as the sum of $\sigma_{pq}^{(m)}(\tau_\alpha)$ and a polynomial in $D^{m-1}\sigma_{pq}(\tau_\alpha)/D\tau_\alpha^{m-1}, \ldots, D\sigma_{pq}(\tau_\alpha)/D\tau_\alpha, \sigma_{pq}(\tau_\alpha), \partial v_p^{(m)}(\tau_\alpha)/\partial x_q(\tau_\alpha), \ldots, \partial v_p^{(1)}(\tau_\alpha)/\partial x_q(\tau_\alpha)$, with similar results obtained by replacing m by $m-1$, $m-2$, ..., Hence, equations (5.1) can be written in the different form

$$\begin{aligned}
f_{ij}\Bigg[&\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, \frac{\partial v_p^{(1)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}, \ldots, \frac{\partial v_p^{(n)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}; \\
&\sigma_{pq}(\tau_\alpha), \sigma_{pq}^{(1)}(\tau_\alpha), \ldots, \sigma_{pq}^{(m)}(\tau_\alpha)\Bigg] = 0 \qquad (n \geqq m),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
f_{ij}\Bigg[&\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, \frac{\partial v_p^{(1)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}, \ldots, \frac{\partial v_p^{(m)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}; \\
&\sigma_{pq}(\tau_\alpha), \sigma_{pq}^{(1)}(\tau_\alpha), \ldots, \sigma_{pq}^{(m)}(\tau_\alpha)\Bigg] = 0 \qquad (n \leqq m).
\end{aligned} \tag{5.3}$$

* In order to specify the constitutive equation completely, we must in general state appropriate initial conditions on the differential equations (5.1).

Moreover if (5.1) is a polynomial in its arguments so also are (5.2) and (5.3). Without loss of generality we may consider equations of the form (5.2) where n, m are integers such that $n \geqq m$.

With the help of (3.6) and (4.19) we may express $2\partial v_p^{(n)}(\tau_\alpha)/\partial x_q(\tau_\alpha)$ as the sum of $2\omega_{pq}^{(n)}(\tau_\alpha)+A_{pq}^{(n)}(\tau_\alpha)$ and a polynomial in the gradients $\partial v_p^{(r)}(\tau_\alpha)/\partial x_q(\tau_\alpha)$ for $r=n-1, n-2, \ldots, 1$. Similarly $2\partial v_p^{(n-1)}(\tau_\alpha)/\partial x_q(\tau_\alpha)$ can be replaced by the sum of $2\omega_{pq}^{(n-1)}(\tau_\alpha)+A_{pq}^{(n-1)}(\tau_\alpha)$ and a polynomial in the gradients $\partial v_p^{(r)}(\tau_\alpha)/\partial x_q(\tau_\alpha)$ for $r=n-2, n-3, \ldots, 1$, and so on. Hence, by successive replacement, we can reduce (5.2) to the different form

$$
\begin{aligned}
f_{ij}\Bigg[&\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, A_{pq}^{(1)}(\tau_\alpha), \ldots, A_{pq}^{(n)}(\tau_\alpha);\\
&\sigma_{pq}(\tau_\alpha), \sigma_{pq}^{(1)}(\tau_\alpha), \ldots, \sigma_{pq}^{(m)}(\tau_\alpha); \omega_{pq}^{(1)}(\tau_\alpha), \ldots, \omega_{pq}^{(n)}(\tau_\alpha)\Bigg]=0 \quad (n \geqq m).
\end{aligned} \tag{5.4}
$$

If (5.2) is a polynomial form we see that (5.4) is also a polynomial. Hence, if (5.1) is a polynomial form so also is (5.4). On the other hand if (5.1) is a continuous function of its arguments with continuous derivatives up to some order, so also is (5.4).

We now consider a motion of the body which differs from that described by (2.2) by a superposed rigid rotation. Such a motion is given by (2.7) and (2.2). Now suppose that at the times τ_α ($\alpha=0, 1, \ldots, N$) the body is in the same instantaneous position as that given by (2.2) at these times, but that it has superposed angular velocity, angular acceleration, ..., $(n-1)^{\text{th}}$ angular acceleration on the motion of every point of the body. Hence $a_{ij}(\tau_\alpha)=\delta_{ij}$ ($\alpha=0, 1, \ldots, N$) and in view of (4.22) equations (5.4) are replaced by

$$
\begin{aligned}
f_{ij}\Bigg[&\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, A_{pq}^{(1)}(\tau_\alpha), \ldots, A_{pq}^{(n)}(\tau_\alpha);\\
&\sigma_{pq}(\tau_\alpha), \sigma_{pq}^{(1)}(\tau_\alpha), \ldots, \sigma_{pq}^{(m)}(\tau_\alpha); \overline{\omega}_{pq}^{(1)}(\tau_\alpha), \ldots, \overline{\omega}_{pq}^{(n)}(\tau_\alpha)\Bigg]=0 \quad (n \geqq m),
\end{aligned} \tag{5.5}
$$

where $\overline{\omega}_{pq}^{(1)}(\tau_\alpha), \ldots, \overline{\omega}_{pq}^{(n)}(\tau_\alpha)$ are given by (3.9). If the form of the constitutive equations is to be independent of the three components of the $(n-1)^{\text{th}}$ angular acceleration of the superposed rigid body rotation, at each of the times τ_α, then since the $(n-1)^{\text{th}}$ angular accelerations are independent at each of these times $\omega_{pq}^{(n)}(\tau_\alpha)$ ($\alpha=0, 1, \ldots, N$) cannot occur explicitly in (5.4). If $\omega_{pq}^{(n)}(\tau_\alpha)$ is removed from (5.4) and $\overline{\omega}_{pq}^{(n)}(\tau_\alpha)$ is removed from (5.5) then the form of the constitutive equations will be independent of the $(n-2)^{\text{th}}$ angular acceleration of the superposed rigid body rotation at each of the times τ_α if $\omega_{pq}^{(n-1)}(\tau_\alpha)$ does not occur in (5.4). Hence, by successive reductions of this type, we see that the equations (5.5) will be independent of any angular velocity, angular acceleration, ..., $(n-1)^{\text{th}}$ angular acceleration of the rigid body rotation which may be superposed at times τ_α, the body occupying the same instantaneous positions at these times, if the equations (5.4) reduce to

$$
\begin{gathered}
f_{ij}\left[\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, A_{pq}^{(1)}(\tau_\alpha), \ldots, A_{pq}^{(n)}(\tau_\alpha); \sigma_{pq}(\tau_\alpha), \sigma_{pq}^{(1)}(\tau_\alpha), \ldots, \sigma_{pq}^{(m)}(\tau_\alpha)\right]=0,\\
(n \geqq m).
\end{gathered} \tag{5.6}
$$

So far our considerations have been purely formal. We shall now show how they can be applied in particular cases.

6. Functional Dependence of Stress on Kinematic Variables

Let us now suppose that the stress components σ_{ij} at time t are functionals of the gradients $\partial x_p(\tau)/\partial X_q$, $\partial x_p(\tau)/\partial x_q$, $\partial v_p^{(1)}(\tau)/\partial x_q(\tau)$, $\partial v_p^{(2)}(\tau)/\partial x_q(\tau)$, ..., $\partial v_p^{(n)}(\tau)/\partial x_q(\tau)$, thus:

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}, \frac{\partial x_p(\tau)}{\partial x_q}, \frac{\partial v_p^{(1)}(\tau)}{\partial x_q(\tau)}, \ldots, \frac{\partial v_p^{(n)}(\tau)}{\partial x_q(\tau)}\right]. \tag{6.1}$$

We first replace the functional F_{ij} by a polynomial function f_{ij} of the values of the arguments at $N+1$ discrete instants of time τ_α ($\alpha=0, 1, 2, \ldots, N$), where $\tau_0=t$, and obtain

$$\sigma_{ij} = f_{ij}\left[\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, \frac{\partial v_p^{(1)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}, \ldots, \frac{\partial v_p^{(n)}(\tau_\alpha)}{\partial x_q(\tau_\alpha)}\right]. \tag{6.2}$$

Following the argument of § 5, we see that the relation (6.2) must be expressible in the form

$$\sigma_{ij} = f_{ij}\left[\frac{\partial x_p(\tau_\alpha)}{\partial X_q}, \frac{\partial x_p(\tau_\alpha)}{\partial x_q}, A_{pq}^{(1)}(\tau_\alpha), \ldots, A_{pq}^{(n)}(\tau)\right], \tag{6.3}$$

where the polynomial function f_{ij} is not in general the same as that in (6.2).

We now let $N+1\to\infty$, so that the function f_{ij} in (6.3) becomes a functional. We then have

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}, \frac{\partial x_p(\tau)}{\partial x_q}, A_{pq}^{(1)}(\tau), \ldots, A_{pq}^{(n)}(\tau)\right], \tag{6.4}$$

where the functional F_{ij} is not in general the same as that in (6.1)*. From (4.20), we have

$$\frac{D^s g_{mn}(\tau)}{D\tau^s} = \frac{\partial x_i(\tau)}{\partial X_m}\frac{\partial x_j(\tau)}{\partial X_n} A_{ij}^{(s)}(\tau)$$

and

$$A_{ij}^{(s)}(\tau) = \frac{\partial X_m}{\partial x_i(\tau)}\frac{\partial X_n}{\partial x_j(\tau)}\frac{D^s g_{mn}(\tau)}{D\tau^s}. \tag{6.5}$$

We note also that

$$\frac{\partial x_p(\tau)}{\partial x_q} = \frac{\partial x_p(\tau)}{\partial X_r}\frac{\partial X_r}{\partial x_q} \tag{6.6}$$

and

$$\frac{\partial X_r}{\partial x_q} = \frac{1}{2g^{\frac{1}{2}}} e_{rkl} e_{qmn} \frac{\partial x_k}{\partial X_m}\frac{\partial x_l}{\partial X_n}. \tag{6.7}$$

From (6.6) and (6.7) we see that any constitutive equation of the form (6.4) is equivalent to a constitutive equation of the form

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}, A_{pq}^{(1)}(\tau), \ldots, A_{pq}^{(n)}(\tau)\right]. \tag{6.8}$$

From (6.5), we note that (6.8) may be expressed in the form

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}, \frac{D g_{pq}(\tau)}{D\tau}, \ldots, \frac{D^n g_{pq}(\tau)}{D\tau^n}\right]. \tag{6.9}$$

---

* We shall use F_{ij} to denote a functional generically in many places below. They will not, in general, all denote the same functionals of the indicated arguments.

We note that if in (6.4) F_{ij} is a continuous functional* of the argument functions in the interval $0 \leqq \tau < t$ and particularly dependent on their values at $\tau = t$, then so, in general, is the functional F_{ij} in (6.9). We shall assume that the particular dependence on the values of the argument functions at $\tau = t$ is sufficiently regular so that F_{ij} in (6.9) may be expressed as a polynomial in $\partial x_p/\partial X_q$, $Dg_{pq}/Dt, \ldots, D^n g_{pq}/Dt^n$, the coefficients in this polynomial being continuous functionals of the argument functions over the range $0 \leqq \tau \leqq t$, and the notation $g_{pq} = g_{pq}(t)$ is used. By a procedure similar to that used in a slightly different context in previous papers [1, 2, 3], each of the coefficient functionals may be uniformly approximated by the sum of a number of terms of the form

$$\begin{aligned} \int_0^t \int_0^t \cdots \int_0^t \chi_{i j i_1 j_1 i_2 j_2 \ldots p_1 q_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R) \frac{\partial x_{p_1}(\tau_1)}{\partial X_{q_1}} \frac{\partial x_{p_2}(\tau_2)}{\partial X_{q_2}} \cdots \\ \frac{D g_{p_K q_K}(\tau_K)}{D \tau_K} \cdots \frac{D^2 g_{p_L q_L}(\tau_L)}{D \tau_L^2} \cdots \frac{D^n g_{p_R q_R}(\tau_R)}{D \tau_R^n} d\tau_1 d\tau_2 \ldots d\tau_R, \end{aligned} \tag{6.10}$$

where the χ's are continuous functions of their arguments.

Now

$$\begin{aligned} &\int_0^t \chi_{i j i_1 j_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R) \frac{D^s g_{p_N q_N}(\tau_N)}{D \tau_N^s} d\tau_N \\ &\quad = \int_0^t \frac{D}{D\tau_N} \left\{ \chi_{i j i_1 j_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R) \frac{D^{s-1} g_{p_N q_N}(\tau_N)}{D \tau_N^{s-1}} \right\} d\tau_N - \\ &\quad\quad - \int_0^t \frac{D \chi_{i j i_1 j_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R)}{D\tau_N} \frac{D^{s-1} g_{p_N q_N}(\tau_N)}{D \tau_N^{s-1}} d\tau_N \\ &\quad = \left[\chi_{i j i_1 j_1 \ldots p_R q_R}(t, \tau_1, \ldots, \tau_R) \frac{D^{s-1} g_{p_N q_N}(\tau_N)}{D \tau_N^{s-1}} \right]_{\tau_N = 0}^t - \\ &\quad\quad - \int_0^t \frac{D \chi_{i j i_1 j_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R)}{D\tau_N} \frac{D^{s-1} g_{p_N q_N}(\tau_N)}{D \tau_N^{s-1}} d\tau_N. \end{aligned} \tag{6.11}$$

We assume that the motion starts sufficiently smoothly so that when $\tau = 0^+$, $v_p^{(s)}(\tau)$ $(s = 1, 2, \ldots, n-1)$ vanish and hence $D^s g_{pq}(\tau)/D\tau^s$ $(s = 1, 2, \ldots, n-1)$ vanish. Then, provided the functions involved are sufficiently regular so that the differentiations may be carried out, we see that we may, without further loss of generality, replace the functionals (6.10) by functionals which are continuous functionals of $\partial x_p(\tau)/\partial X_q$ and polynomial functions of $\partial x_p/\partial X_q$, Dg_{pq}/Dt, $\ldots, D^n g_{pq}/Dt^n$ and hence of $\partial x_p/\partial X_q$, $A_{pq}^{(1)}, \ldots, A_{pq}^{(n)}$. Thus, in (6.8), we may take σ_{ij} to be a polynomial function of $\partial x_p/\partial X_q$, $A_{pq}^{(1)}, \ldots, A_{pq}^{(n)}$ and a continuous functional of $\partial x_p(\tau)/\partial X_q$, so that

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t} \left[\frac{\partial x_p(\tau)}{\partial X_q}; \frac{\partial x_p}{\partial X_q}, A_{pq}^{(1)}, \ldots, A_{pq}^{(n)} \right]. \tag{6.12}$$

* Throughout this paper, where we describe a functional as a continuous functional, we shall mean that it is a continuous functional over the compact aggregate of the argument functions which are continuous in the range $0 \leqq \tau \leqq t$.

Similarly, in (6.9), we may take σ_{ij} to be a polynomial function of $\partial x_p/\partial X_q$, $Dg_{pq}/Dt, \ldots, D^n g_{pq}/Dt^n$ and a continuous functional of $\partial x_p(\tau)/\partial X_q$, so that

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}; \frac{\partial x_p}{\partial X_q}, \frac{Dg_{pq}}{Dt}, \ldots, \frac{D^n g_{pq}}{Dt^n}\right]. \tag{6.13}$$

In the next section we shall consider constitutive equations of the types (6.12) and (6.13) further.

The results (6.12) and (6.13) may be obtained by a slightly different procedure. As before, we see that we may without loss of generality omit the argument functions $\partial x_p(\tau)/\partial x_q$ in (6.1). We then assume that in (6.1) σ_{ij} is a continuous functional of the argument functions in the range $0 \leq \tau < t$ and is particularly dependent on their values at $\tau = t$. Using the relation (6.7) it follows that σ_{ij} is a continuous functional of $\partial x_p(\tau)/\partial X_q$, $\partial v_p^{(1)}(\tau)/\partial X_q, \ldots, \partial v_p^{(n)}(\tau)/\partial X_q$ in the range $0 \leq \tau < t$ and an ordinary function of the values of these functions at $\tau = t$. We assume sufficient regularity so that σ_{ij} may be uniformly approximated by a polynomial in $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial X_q, \ldots, \partial v_p^{(n)}/\partial X_q$, the coefficients in which are continuous functionals of $\partial x_p(\tau)/\partial X_q$, $\partial v_p^{(1)}(\tau)/\partial X_q, \ldots, \partial v_p^{(n)}(\tau)/\partial X_q$. A typical term in the expression for σ_{ij} is then

$$\begin{gathered}\frac{\partial v_{i_1}^{(\alpha_1)}}{\partial X_{j_1}} \frac{\partial v_{i_2}^{(\alpha_2)}}{\partial X_{j_2}} \cdots \frac{\partial v_{i_R}^{(\alpha_R)}}{\partial X_{j_R}} \int_0^t \cdots \int_0^t \chi_{ij i_1 j_1 \ldots i_R j_R p_1 q_1 \ldots p_S q_S}(t, \tau_1, \tau_2, \ldots, \tau_S) \times \\ \times \frac{\partial v_{p_1}^{(\beta_1)}(\tau_1)}{\partial X_{q_1}} \frac{\partial v_{p_2}^{(\beta_2)}(\tau_2)}{\partial X_{q_2}} \cdots \frac{\partial v_{p_S}^{(\beta_S)}(\tau_S)}{\partial X_{q_S}} d\tau_1 d\tau_2 \ldots d\tau_S,\end{gathered} \tag{6.14}$$

where the kernels are continuous functions of their arguments, $\alpha_1, \alpha_2, \ldots, \alpha_R$ and $\beta_1, \beta_2, \ldots, \beta_S$ are chosen from the integers $0, 1, 2, \ldots, n$ and $v_i^{(0)}$ denotes x_i.

We note that

$$\begin{aligned}&\int \chi_{ij i_1 j_1 \ldots p_S q_S}(t, \tau_1, \ldots, \tau_S) \frac{\partial v_{p_N}^{(\beta_N)}(\tau_N)}{\partial X_{q_N}} d\tau_N \\ &= \left[\chi_{ij i_1 j_1 \ldots p_S q_S} \frac{\partial v_{p_N}^{(\beta_N - 1)}(\tau_N)}{\partial X_{q_N}}\right]_{\tau_N=0}^{t} - \int_0^t \frac{D\chi_{ij i_1 j_1 \ldots p_S q_S}}{D\tau_N} \frac{\partial v_{p_N}^{(\beta_N - 1)}(\tau_N)}{\partial X_{q_N}} d\tau_N.\end{aligned} \tag{6.15}$$

By repeated use of relations of this type it is seen that σ_{ij} may be uniformly approximated by a polynomial function of $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial X_q, \ldots, \partial v_p^{(n)}/\partial X_q$, the coefficients in which are continuous functionals of $\partial x_p(\tau)/\partial X_q$. Thus,

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}; \frac{\partial x_p}{\partial X_q}, \frac{\partial v_p^{(1)}}{\partial X_q}, \ldots, \frac{\partial v_p^{(n)}}{\partial X_q}\right]. \tag{6.16}$$

Using the relation (6.7) it is seen that σ_{ij} may be uniformly approximated by a polynomial function of $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$, the coefficients in which are continuous functionals of $\partial x_p(\tau)/\partial X_q$. Thus,

$$\sigma_{ij} = \mathop{F_{ij}}_{\tau=0}^{t}\left[\frac{\partial x_p(\tau)}{\partial X_q}; \frac{\partial x_p}{\partial X_q}, \frac{\partial v_p^{(1)}}{\partial x_q}, \ldots, \frac{\partial v_p^{(n)}}{\partial x_q}\right]. \tag{6.17}$$

In a manner similar to that used in deriving (6.12) from (6.1) we can show that (6.17) must be expressible in the form (6.12).

7. Further Restrictions on the Constitutive Equation

We take as our starting point a constitutive equation of the form (6.13) and investigate the further restrictions which may be imposed on it, short of assuming that the material has some symmetry.

From (4.17), we have with the notation $t^{mn}(\tau)=t^{mn}$

$$\sigma_{ij}=\frac{\partial x_i}{\partial X_m}\frac{\partial x_j}{\partial X_n}t^{mn}$$

and

$$t^{mn}=\frac{\partial X_m}{\partial x_i}\frac{\partial X_n}{\partial x_j}\sigma_{ij}. \tag{7.1}$$

It follows from (6.13), (7.1) and (6.7) that

$$t^{ij}=\underset{\tau=0}{\overset{t}{F_{ij}}}\left[\frac{\partial x_p(\tau)}{\partial X_q};\frac{\partial x_p}{\partial X_q},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right], \tag{7.2}$$

where F_{ij} denotes a different dependence on the arguments from that in (6.13).

We replace the functional dependence of t^{ij} on $\partial x_p(\tau)/\partial X_q$ by polynomial dependence f_{ij} on the values of the functions at $N+1$ discrete instants of time τ_α $(\alpha=0, 1, 2, \ldots, N+1)$, with $\tau_0=t$, thus:

$$t^{ij}=f_{ij}\left[\frac{\partial x_p(\tau_\alpha)}{\partial X_q},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right]. \tag{7.3}$$

We next consider a deformation of the body which differs from that described by (2.2) only by an arbitrary rigid-body rotation (2.7) in a small time interval about τ_β. From § 4, we see that t^{ij}, $Dg_{pq}/Dt, \ldots, D^n g_{pq}/Dt^n$ and $\partial x_p(\tau_\alpha)/\partial X_q$, $(\tau_\alpha \neq \tau_p)$, are unaltered by this rotation, so that using (7.2) we have

$$\begin{aligned} f_{ij}&\left[\frac{\partial \bar{x}_p(\tau_\beta)}{\partial X_q},\frac{\partial x_p(\tau_\alpha)}{\partial X_q},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right]\\ &=f_{ij}\left[\frac{\partial x_p(\tau_\beta)}{\partial X_q},\frac{\partial x_p(\tau_\alpha)}{\partial X_q},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right].\end{aligned} \tag{7.4}$$

Thus, f_{ij} is a polynomial scalar invariant under proper orthogonal transformations of the vectors $\partial x_p(\tau_\beta)/\partial X_1$, $\partial x_p(\tau_\beta)/\partial X_2$ and $\partial x_p(\tau_\beta)/\partial X_3$. Hence we see that f_{ij} must be expressible as a polynomial in $g_{pq}(\tau_\beta)$ and $[g(\tau_\beta)]^{\frac{1}{2}}$. Repeating this procedure for each value of τ_α in turn, we see that

$$t^{ij}=f_{ij}\left[g_{pq}(\tau_\alpha),\{g(\tau_\alpha)\}^{\frac{1}{2}},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right]. \tag{7.5}$$

Now, allowing $N+1\to\infty$, we obtain

$$t^{ij}=\underset{\tau=0}{\overset{t}{F_{ij}}}\left[g_{pq}(\tau);g_{pq},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right]. \tag{7.6}$$

From (7.1) and (7.6), we obtain

$$\sigma_{ij}=\frac{\partial x_i}{\partial X_m}\frac{\partial x_j}{\partial X_n}\underset{\tau=0}{\overset{t}{F_{mn}}}\left[g_{pq}(\tau);g_{pq},\frac{D g_{pq}}{D t},\ldots,\frac{D^n g_{pq}}{D t^n}\right]. \tag{7.7}$$

Using (6.5), we may re-write (7.7) in the form

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_n} F_{mn} \Big[\underset{\tau=0}{\overset{t}{g_{pq}(\tau)}} ; g_{pq}, \frac{\partial x_r}{\partial X_p} \frac{\partial x_s}{\partial X_q} A^{(1)}_{rs}, \ldots, \frac{\partial x_r}{\partial X_p} \frac{\partial x_s}{\partial X_q} A^{(n)}_{rs} \Big]. \tag{7.8}$$

In (7.6), (7.7) and (7.8), F_{ij} is a continuous functional of the argument functions $g_{pq}(\tau)$ and particularly dependent on the remaining arguments.

8. Some Particular Cases

First we shall consider the case in which we assume in (6.13) that σ_{ij} is independent of $Dg_{pq}/Dt, \ldots, D^n g_{pq}/Dt^n$. Accordingly, we take $Dg_{pq}/Dt = D^2 g_{pq}/Dt^2 = \cdots = D^n g_{pq}/Dt^n = 0$ in (7.7). We see that in this case the constitutive equation becomes

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_n} F_{mn} [\underset{\tau=0}{\overset{t}{g_{pq}(\tau)}} ; g_{pq}]. \tag{8.1}$$

This result has already been obtained in previous papers [*1*, *3*].

Next, we consider the case when, in (6.13), σ_{ij} is an ordinary function of $\partial x_p/\partial X_q, Dg_{pq}/Dt, \ldots, D^n g_{pq}/Dt^n$, and hence of $\partial x_p/\partial X_q, A^{(1)}_{pq}, \ldots, A^{(n)}_{pq}$. This arises from the assumption in (6.1) that σ_{ij} is an ordinary function of the gradients $\partial x_p/\partial X_q, \partial v^{(1)}_p/\partial x_q, \ldots, \partial v^{(n)}_p/\partial x_q$. We then have from (7.7) and (7.8)

$$\begin{aligned} \sigma_{ij} &= \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_n} f_{mn} \left(g_{pq}, \frac{D g_{pq}}{Dt}, \ldots, \frac{D^n g_{pq}}{Dt^n} \right) \\ &= \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_n} f_{mn} \left(g_{pq}, \frac{\partial x_r}{\partial X_p} \frac{\partial x_s}{\partial X_q} A^{(1)}_{rs}, \ldots, \frac{\partial x_r}{\partial X_p} \frac{\partial x_s}{\partial X_q} A^{(n)}_{rs} \right). \end{aligned} \tag{8.2}$$

We now consider the further particularization in which F_{ij} in (6.1) and (7.2) are independent of $\partial x_p/\partial X_q$, so that in (8.2) σ_{ij} depends on $A^{(1)}_{pq}, A^{(2)}_{pq}, \ldots, A^{(n)}_{pq}$ only. The particularization can be made formally in (6.1) by taking $\partial x_i/\partial X_j = \delta_{ij}$. We therefore introduce this substitution into (8.2) and obtain

$$\sigma_{ij} = f_{ij} (A^{(1)}_{pq}, A^{(2)}_{pq}, \ldots, A^{(n)}_{pq}). \tag{8.3}$$

This result has been previously obtained by RIVLIN & ERICKSEN [*6*]. Alternatively, we may derive this result by assuming that the stress components σ_{ij} at time t, in the rectangular Cartesian coordinate system x, depend only on the gradients of the velocity, acceleration, ... in the system x. Equation (8.3) may then be derived by the method employed in obtaining (5.6) from (5.1).

The particularization of the constitutive equation to the form (8.3) implies that the material is initially isotropic as has, in essence, been pointed out by NOLL [*7*]. In [*6*] and later papers [*8*, *9*, *10*, *11*] further implications of this isotropy with regard to the form of the constitutive equation have been explored.

As a final special case that merits attention, we consider that in (6.12), the particular dependence of F_{ij} on $\partial x_p/\partial X_q$ is of such a nature that the equation may be written in the form

$$\sigma_{ij} = F_{ij} \Big[\underset{\tau=0}{\overset{t}{\frac{\partial x_p(\tau)}{\partial x_q}}} ; A^{(1)}_{pq}, A^{(2)}_{pq}, \ldots, A^{(n)}_{pq} \Big]. \tag{8.4}$$

This constitutive equation can be obtained formally from the more general constitutive equation (6.12) by taking $\partial x_p/\partial X_q = \delta_{pq}$. Accordingly, introducing this into (7.8) to eliminate $\partial/\partial X_p$, we see that (7.8) becomes

$$\sigma_{ij} = \underset{\tau=0}{\overset{t}{F_{ij}}}[\gamma_{pq}(\tau);\, A^{(1)}_{pq}, A^{(2)}_{pq}, \ldots, A^{(n)}_{pq}], \tag{8.5}$$

where

$$\gamma_{pq}(\tau) = \frac{\partial x_m(\tau)}{\partial x_p}\,\frac{\partial x_m(\tau)}{\partial x_q}. \tag{8.6}$$

We thus conclude that a constitutive equation of the form (8.4) can be written in the form (8.5), where F_{ij} is a continuous functional of $\gamma_{pq}(\tau)$ and an ordinary function of $A^{(1)}_{pq}, \ldots, A^{(n)}_{pq}$.

This result can, of course, be obtained by assuming that in a specified coordinate system x, the constitutive equation has the form

$$\sigma_{ij} = \underset{\tau=0}{\overset{t}{F_{ij}}}\left[\frac{\partial x_p(\tau)}{\partial x_q};\, \frac{\partial v^{(1)}_p}{\partial x_q}, \frac{\partial v^{(2)}_p}{\partial x_q}, \ldots, \frac{\partial v^{(n)}_p}{\partial x_q}\right]. \tag{8.7}$$

Then, in the manner adopted in § 5, we find that it must be expressible in the form (8.4). We now consider successively deformations which differ from the assumed deformation by a rigid rotation at each instant of time except the instant t. Since the stress components in the system x are unaltered by such rotations, we readily obtain the form (8.5) for the constitutive equation.

The particularization of the constitutive equation to the form (8.5) implies that the material is initially isotropic, from considerations similar to those used by Noll [7]. The further implications of this isotropy with regard to the form of the constitutive equation can be analysed in a manner similar to that adopted in [2] in a slightly different context.

In the case when the dependence on $A^{(1)}_{pq}, \ldots, A^{(n)}_{pq}$ is omitted, (8.5) takes the form

$$\sigma_{ij} = \underset{\tau=0}{\overset{t}{F_{ij}}}[\gamma_{rs}(\tau)]. \tag{8.8}$$

This form of constitutive equation has been particularly discussed by Noll [5]. It may also be derived from (8.1) in a manner similar to that used in deriving (8.5) from (7.8).

9. Alternative Formulation of Particular Dependence

We have seen that if we take as our starting point a constitutive equation of the type (6.1), there is no substantial loss of generality in considering that the constitutive equation has the form (6.16). Let us assume for simplicity that, in (6.16), the dependence of σ_{ij} on $\partial x_p/\partial X_q$, $\partial v^{(1)}_p/\partial X_q, \ldots, \partial v^{(n)}_p/\partial X_q$ is polynomial dependence, the coefficients in the polynomial being continuous functionals of $\partial x_p(\tau)/\partial X_q$. Then a typical term in the expression for σ_{ij} is

$$\begin{aligned}\frac{\partial v^{(\alpha)}_{i_1}}{\partial X_{j_1}}\,\frac{\partial v^{(\beta)}_{i_2}}{\partial X_{j_2}}\cdots\frac{\partial v^{(\gamma)}_{i_R}}{\partial X_{j_R}}\int_0^t\cdots\int_0^t \chi_{i j i_1 j_1 \ldots i_R j_R p_1 q_1 \ldots p_S q_S}(t, \tau_1, \tau_2, \ldots, \tau_S)\,\times \\ \times\,\frac{\partial x_{p_1}(\tau_1)}{\partial X_{q_1}}\,\frac{\partial x_{p_2}(\tau_2)}{\partial X_{q_2}}\cdots\frac{\partial x_{p_S}(\tau_S)}{\partial X_{q_S}}\,d\tau_1\,d\tau_2\ldots d\tau_S,\end{aligned} \tag{9.1}$$

where the kernels are continuous functions of their arguments, $\alpha, \beta, \ldots, \gamma$ are chosen from the integers 0, 1, 2, ..., n and $v_i^{(0)}$ denotes x_i, provided the motion starts sufficiently smoothly.

Denoting the Dirac delta function and its derivatives by $\delta(\tau)$, $\delta'(\tau)$, ..., $\delta^{(s)}(\tau)$, ..., we have

$$\frac{\partial v_p^{(0)}}{\partial X_q} = \frac{\partial x_p}{\partial X_q} = \int_0^t \delta(\tau - t) \frac{\partial x_p(\tau)}{\partial X_q} d\tau,$$

and (9.2)

$$\frac{\partial v_p^{(s)}}{\partial X_q} = \int_0^t \delta^{(s)}(\tau - t) \frac{\partial x_p(\tau)}{\partial X_q} d\tau \qquad (s = 1, 2, \ldots, n).$$

Using these relations it is apparent that we may omit the explicit dependence of the functionals on $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial X_q, \ldots, \partial v_p^{(n)}/\partial X_q$ in (6.16) and on $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$ in (6.17) if we allow the kernels to depend on the Dirac delta function and its derivatives at time t.

10. Relation to a Previous Theory

Rivlin & Ericksen [6] have investigated the implications of an initial assumption that the constitutive equation takes the form of a relation between the stress components σ_{ij} and the kinematic gradients $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. It is easy to show that in a wide variety of cases the assumption of a constitutive equation of the form (6.17) is equivalent to the assumption made by Rivlin & Ericksen.

We assume in (6.17) that the functional dependence is continuous in the range $0 \leqq \tau < t$ and that σ_{ij} is particularly dependent on the values of the gradients at $\tau = t$. We also assume sufficient regularity so that σ_{ij} may be uniformly approximated by the sum of a number of terms of the form

$$\int_0^t \int_0^t \cdots \int_0^t \chi_{i j i_1 j_1 \ldots i_M j_M p_1 q_1 \ldots p_R q_R}(t, \tau_1, \tau_2, \ldots, \tau_R) \times \\ \times \frac{\partial x_{p_1}(\tau_1)}{\partial X_{q_1}} \frac{\partial x_{p_2}(\tau_2)}{\partial X_{q_2}} \cdots \frac{\partial x_{p_R}(\tau_R)}{\partial X_{q_R}} d\tau_1 d\tau_2 \ldots d\tau_R \tag{10.1}$$

the coefficients of which are ordinary functions of $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. It is easy to show that if the functionals are of the hereditary type, the typical term (10.1) must take the form

$$\int_{t-T}^t \int_{t-T}^t \cdots \int_{t-T}^t \chi_{i j i_1 j_1 \ldots i_M j_M p_1 q_1 \ldots p_R q_R}(t - \tau_1, t - \tau_2, \ldots, t - \tau_R) \times \\ \times \frac{\partial x_{p_1}(\tau_1)}{\partial X_{q_1}} \frac{\partial x_{p_2}(\tau_2)}{\partial X_{q_2}} \cdots \frac{\partial x_{p_R}(\tau_R)}{\partial X_{q_R}} d\tau_1 d\tau_2 \ldots d\tau_R, \tag{10.2}$$

where T is a time such that $\chi_{i \ldots q_R}$ vanishes if any of the variables $\tau_1, \ldots, \tau_R < t - T$.

Now, if in (2.2) x_p is a polynomial function* of τ, it may be expressed as a polynomial function of $t - \tau$, thus:

$$x_p(\tau) = x_p + (t - \tau) v_p^{(1)} + \frac{1}{2!} (t - \tau)^2 v_p^{(2)} + \cdots. \tag{10.3}$$

* This cannot, of course, be the case if $T > t$.

Introducing (10.3) into (10.2) it is apparent that (10.2) may be expressed as a polynomial in $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(K)}/\partial x_q$. Hence, the stress components may be expressed as functions of these gradients.

More generally, if x_p may be expressed as a Taylor series★, about the instant of time t, by an expression of the form (10.3) together with a remainder term, and if this series converges throughout the interval $t-T \leqq \tau \leqq t$, then we have the result that the stress components may be uniformly approximated by functions of $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$.

If we do not make the assumption that the functionals are of the hereditary type, then if $x_p(\tau)$ may be expressed as a Taylor series which is uniformly convergent in the range $0 \leqq \tau \leqq t$, σ_{ij} may be uniformly approximated by a function of $\partial x_p/\partial X_q$, $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$ and t.

In a similar way we can show that an assumption that the constitutive equation takes the form

$$\sigma_{ij} = \underset{\tau=0}{\overset{t}{F_{ij}}}\left[\frac{\partial x_p(\tau)}{\partial x_q}; \frac{\partial v_p^{(1)}}{\partial x_q}, \ldots, \frac{\partial v_p^{(n)}}{\partial x_q}\right], \tag{10.4}$$

is, in a wide variety of cases, equivalent to an initial assumption that the stress components are functions of the kinematic gradients $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. Again, assuming that the dependence of σ_{ij} on the kinematic variables is sufficiently regular and of the hereditary type, we may express σ_{ij} as the sum of a number of terms of the type

$$\begin{aligned}\int_{t-T}^{t} \int_{t-T}^{t} \cdots \int_{t-T}^{t} \chi_{i j i_1 j_1 \ldots i_M j_M p_1 q_1 \ldots p_R q_R}(t-\tau_1, t-\tau_2, \ldots, t-\tau_R) \times \\ \times \frac{\partial x_{p_1}(\tau_1)}{\partial x_{q_1}} \cdots \frac{\partial x_{p_R}(\tau_R)}{\partial x_{q_R}} d\tau_1 \ldots d\tau_R,\end{aligned} \tag{10.5}$$

with coefficients which are functions of $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. In (10.5), T is a time such that $\chi_{i \ldots q_R}$ vanishes if any of the variables $\tau_1, \tau_2, \ldots, \tau_R < t-T$. Again we assume, as in the case previously considered, that $x_p(\tau)$ can be expanded as a Taylor series about $\tau-t$, which converges throughout the interval $t-T \leqq \tau \leqq t$, or as a special case can be expressed as a polynomial in τ. It is then seen that (10.5) and hence σ_{ij} may be uniformly approximated by a function of $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$.

Coleman & Noll [*12*] have calculated, on the basis of a "memory" theory the forces necessary to support certain simple states of flow in an isotropic visco-elastic fluid. These types of flow had previously been discussed [*13*] on the assumption that the constitutive equation takes the form, advanced by Rivlin & Ericksen [*6*], of a relation between the stress components σ_{ij} and the kinematic gradients $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. It is easy to see that for these flow fields the "memory" theory is equivalent to a theory of the type previously advanced by Rivlin & Ericksen.

In the case of rectilinear flow through a straight tube of circular cross-section, taking the x_3 axis parallel to the length of the tube, we have

$$x_1(\tau) = x_1, \quad x_2(\tau) = x_2, \quad x_3(\tau) = x_3 - (t-\tau)\, v_3^{(1)}, \tag{10.6}$$

★ In the case when $T > t$, this restricts the flow to start at $\tau = 0$ sufficiently smoothly.

where $v_3^{(1)}$ is the velocity along the tube of a particle of the fluid. $v_3^{(1)}$ is, of course, independent of τ but dependent on x_1 and x_2. We note that in this case $x_3(\tau)$ is a polynomial in $t-\tau$.

In the remaining two cases discussed by Coleman & Noll [12], Couette flow and torsional flow, it is easily seen that, if the x_3 axis is chosen to coincide with the axis of flow, the path of a generic particle of the fluid is described by

$$\begin{aligned} x_1(\tau) &= x_1\cos\omega(t-\tau) - x_2\sin\omega(t-\tau), \\ x_2(\tau) &= x_1\sin\omega(t-\tau) + x_2\cos\omega(t-\tau), \\ x_3(\tau) &= x_3. \end{aligned} \tag{10.7}$$

ω is independent of τ and is a function of $(x_1^2+x_2^2)^{\frac{1}{2}}$ in the case of Couette flow and of x_3 in the case of torsional flow. We note that the Taylor series expansions about $\tau=t$ for $\cos\omega(t-\tau)$ and $\sin\omega(t-\tau)$ are uniformly convergent for all finite values of $t-\tau$.

It follows that as far as discussion of these problems is concerned, an initial assumption that the constitutive equation for the visco-elastic fluid takes the form (10.4) is equivalent to an assumption that σ_{ij} depends on $\partial v_p^{(1)}/\partial x_q, \ldots, \partial v_p^{(n)}/\partial x_q$. A similar result obtains in the case of helical flow of the visco-elastic fluid discussed in a previous paper [13].

Note: In general, equations (6.12), (6.13), (6.16) and (6.17) also depend explicitly on the time t, except when the functionals in (6.11) and (6.15) are of the invariable hereditary type.

The term hereditary in this and previous papers is restricted to mean hereditary which is of the invariable type (the case of the closed cycle).

Acknowledgement. As far as one of the authors (R.S.R.) is concerned, the results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract Nonr-562(10) with Brown University.

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(Received December 3, 1959)

The Effect of an Aligned Magnetic Field on Oseen Flow of a Conducting Fluid

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Communicated by Hilda Geiringer

Slow flow of a viscous conducting fluid past a sphere is considered. The classical Oseen solution is modified by a magnetic field which, at infinity, is uniform and in the direction of the free stream. If the pressure of the ambient magnetic field is smaller than the dynamical pressure in the free stream, there is no effect on the Oseen value of the drag. If it is greater, however, the drag increases with the strength of the applied magnetic field and the conductivity of the fluid.

1. Introduction

We consider the slow flow of an incompressible, viscous, electrically conducting fluid past a sphere, which, for simplicity, is assumed to have the same magnetic permeability as the fluid. The applied magnetic field is uniform but weak, and is aligned along the free-stream direction.

This problem has been studied by Chester (1957), who neglected the disturbance of the magnetic field and the inertia of the fluid. This amounts to assuming that the Reyneolds number, R, and the magnetic Reynolds number, R_M, are small compared to the Hartmann number, M, though the latter is itself supposed to be small. In this way he found the drag formula

$$\frac{D}{D_S} = 1 + \frac{3}{8} M + \frac{7}{960} M^2 - \frac{43}{7680} M^3 + O(M^4), \tag{1}$$

where $D_S = 6\pi\varrho_0 \nu a U$ is the Stokes value for the drag in the purely viscous case.

The purpose of this paper is to take into account, to some extent, the effects neglected by Chester, by using an Oseen-type approximation in which quadratic terms in the disturbance quantities are neglected in the equations of motion.

As has been found by Carrier & Greenspan (1959) for a plane problem, the ratio $\mu H_0^2/\varrho_0 U^2 = M^2/R R_M$ of magnetic pressure to dynamic pressure is an important parameter in the discussion. For $M^2 < R R_M$, the two wakes which appear both lie downstream of the sphere, while, for $M^2 > R R_M$, one lies upstream; and the drag experienced by the sphere depends in a completely different way on M, R, and R_M in the two cases. Indeed we find

$$\frac{D}{D_S} = 1 + \frac{3}{8} K, \tag{2}$$

where
$$K=\begin{cases} R \quad \text{for} \quad M^2 < R R_M, \\ \dfrac{2M^2+R^2-RR_M}{\sqrt{(R-R_M)^2+4M^2}} \quad \text{for} \quad M^2 > R R_M, \end{cases}$$

and K is correct, in the present approximation, to the first order in M, R, R_M★. Higher-order terms are irrelevant, since it is known [PROUDMAN & PEARSON (1957)] that, in the purely viscous case ($M=R_M=0$), the Oseen theory only gives D/D_S correct to the first order.

For $M=R_M=0$, $K=R$ and we obtain OSEEN's result. When $R=R_M=0$, $K=M$ and (2) gives the first two terms in CHESTER's formula (1); the remaining terms correspond to higher-order terms in K, and are therefore suspect for the reason just given.

The surprising thing about (2) is that the magnetic field has no first-order effect on the drag so long as $M^2 < R R_M$, but that for $M^2 > R R_M$ it does. In the latter case K is always greater than R: as might be expected, the attempt of the magnetic field to inhibit motion across its lines of force, as the fluid passes around the sphere, results in an increase of drag.

Similar results have been found by VAN BLERKOM (1959).

2. The Equations of Motion

The steady motion of an incompressible, viscous, electrically conducting fluid of constant properties is governed by the equations

$$(3)\quad \begin{aligned} &\text{(a)}\quad \boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v} = -\frac{1}{\varrho_0}\operatorname{grad}p + \nu\nabla^2\boldsymbol{v} + \frac{\mu}{\varrho_0}\operatorname{curl}\boldsymbol{H}\times\boldsymbol{H}, \qquad \text{(b)}\quad \operatorname{div}\boldsymbol{v}=0, \\ &\text{(c)}\quad \operatorname{curl}\boldsymbol{H} = \sigma(\boldsymbol{E}+\mu\boldsymbol{v}\times\boldsymbol{H}), \qquad \text{(d)}\quad \operatorname{curl}\boldsymbol{E}=0, \qquad \text{(e)}\quad \operatorname{div}\boldsymbol{H}=0. \end{aligned}$$

For an axially symmetric motion in which the velocity $\boldsymbol{v}$ and magnetic intensity $\boldsymbol{H}$ lie in the meridian plane and are independent of the azimuthal angle, the conduction equation (3c) shows that the electric field $\boldsymbol{E}$ is perpendicular to this plane and also independent of the azimuthal angle. Equation (3d) then shows that $\boldsymbol{E}=0$, if it is to be finite at the axis.

When $\boldsymbol{v}$, $\boldsymbol{r}$, and $\boldsymbol{H}$ are now made dimensionless by referring them to the velocity at infinity U, the radius of the sphere a, and the external field intensity H_0, the equations reduce to

$$(4)\quad \begin{aligned} &\text{(a)}\quad R\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v} = -\operatorname{grad}p + \nabla^2\boldsymbol{v} + \frac{M^2}{R_M}\operatorname{curl}\boldsymbol{H}\times\boldsymbol{H}, \qquad \text{(b)}\quad \operatorname{div}\boldsymbol{v}=0, \\ &\text{(c)}\quad \operatorname{curl}\boldsymbol{H} = R_M\boldsymbol{v}\times\boldsymbol{H}, \qquad \text{(d)}\quad \operatorname{div}\boldsymbol{H}=0. \end{aligned}$$

Here $R=Ua/\nu$ is the Reynolds number, $R_M=Ua\mu\sigma$ is the magnetic Reynolds number, and $M=\mu H_0 a(\sigma/\varrho_0\nu)^{\frac{1}{2}}$ is the Hartmann number; the pressure is now given by $(\varrho_0\nu U/a)p$.

If the x-axis is taken along the axis of symmetry, then at infinity $\boldsymbol{v}=\boldsymbol{H}=\boldsymbol{i}$. Following OSEEN's work in the purely viscous case, we linearize these equations by writing $\boldsymbol{v}+\boldsymbol{i}$ for $\boldsymbol{v}$ and $\boldsymbol{h}+\boldsymbol{i}$ for $\boldsymbol{H}$, and neglecting quadratic terms in $\boldsymbol{v}$

★ In fact, this result is also valid when the permeability of the sphere is different from that of the fluid.

and $\boldsymbol{h}$. From (4a) we then find

(5a)
$$\nabla^2 \boldsymbol{v} - R\frac{\partial \boldsymbol{v}}{\partial x} + \frac{M^2}{R_M}\frac{\partial \boldsymbol{h}}{\partial x} = \operatorname{grad}\left(p + \frac{M^2}{R_M}\boldsymbol{h}\cdot\boldsymbol{i}\right),$$

while the curl of (4c) yields

(5b)
$$\nabla^2 \boldsymbol{h} + R_M\left(\frac{\partial \boldsymbol{v}}{\partial x} - \frac{\partial \boldsymbol{h}}{\partial x}\right) = 0,$$

when (4b) and (4d) are used. This pair of equations may be replaced by an equivalent pair of Oseen equations, so that their solution follows a well-established path.

3. Solution of the Oseen Equations

From (5a) we have

$$\nabla^2\left(p + \frac{M^2}{R_M}\boldsymbol{h}\cdot\boldsymbol{i}\right) = 0;$$

and a particular integral of equations (5) is therefore obtained if we write

$$R_M p + M^2 \boldsymbol{h}\cdot\boldsymbol{i} = (M^2 - R R_M)\frac{\partial\varphi}{\partial x}, \qquad \boldsymbol{v} = \boldsymbol{h} = \operatorname{grad}\varphi,$$

where φ satisfies

(6)
$$\nabla^2\varphi = 0.$$

The solution is completed if we set

$$\boldsymbol{v} = \operatorname{grad}\varphi + \boldsymbol{v}', \qquad \boldsymbol{h} = \operatorname{grad}\varphi + \boldsymbol{h}',$$

where $\boldsymbol{v}', \boldsymbol{h}'$ are solutions of the corresponding homogeneous equations. Two linear combinations of the latter are the Oseen equations

$$\left(\nabla^2 - 2k_1\frac{\partial}{\partial x}\right)(\boldsymbol{v}' + \alpha_1\boldsymbol{h}') = 0, \qquad \left(\nabla^2 - 2k_2\frac{\partial}{\partial x}\right)(\boldsymbol{v}' + \alpha_2\boldsymbol{h}') = 0,$$

where

$$k_1, k_2 = \frac{1}{4}\left[R + R_M \mp \sqrt{(R - R_M)^2 + 4M^2}\right],$$

$$\alpha_1, \alpha_2 = \frac{1}{2R_M}\left[R - R_M \pm \sqrt{(R - R_M)^2 + 4M^2}\right];$$

in addition we clearly have

$$\operatorname{div}(\boldsymbol{v}' + \alpha_1\boldsymbol{h}') = \operatorname{div}(\boldsymbol{v}' + \alpha_2\boldsymbol{h}') = 0.$$

It follows that we may write [Lamb (1911, 1932)]

(7)
$$\begin{aligned}\boldsymbol{v}' + \alpha_1\boldsymbol{h}' &= (\alpha_2 - \alpha_1)\left[\frac{1}{2k_1}\operatorname{grad}\chi_1 - \chi_1\boldsymbol{i}\right],\\ \boldsymbol{v}' + \alpha_2\boldsymbol{h}' &= (\alpha_2 - \alpha_1)\left[\frac{1}{2k_2}\operatorname{grad}\chi_2 - \chi_2\boldsymbol{i}\right],\end{aligned}$$

where

(8)
$$\left(\nabla^2 - 2k_1\frac{\partial}{\partial x}\right)\chi_1 = \left(\nabla^2 - 2k_2\frac{\partial}{\partial x}\right)\chi_2 = 0.$$

This leads to the complete solution

$$
\begin{aligned}
\boldsymbol{v} &= \operatorname{grad}\left(\varphi + \frac{\alpha_2}{2k_1}\chi_1 - \frac{\alpha_1}{2k_2}\chi_2\right) - (\alpha_2\chi_1 - \alpha_1\chi_2)\,\boldsymbol{i}, \\
\boldsymbol{h} &= \operatorname{grad}\left(\varphi - \frac{1}{2k_1}\chi_1 + \frac{1}{2k_2}\chi_2\right) + (\chi_1 - \chi_2)\,\boldsymbol{i}, \\
p &= -R\frac{\partial\varphi}{\partial x} + \frac{M^2}{R_M}\left[\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x} - \frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right) - (\chi_1 - \chi_2)\right].
\end{aligned} \tag{9}
$$

Inside the sphere $\operatorname{curl}\boldsymbol{h} = \operatorname{div}\boldsymbol{h} = 0$, and hence we may set

$$\boldsymbol{h} = \operatorname{grad}\psi, \tag{10}$$

where

$$\nabla^2\psi = 0. \tag{11}$$

The fundamental solutions of (8), from which the general solutions may be constructed (Section 4), are $\exp[-|k_1|r + k_1 x]/r$ and $\exp[-|k_2|r + k_2 x]/r$, respectively. These lead to wakes in the directions of increasing $k_1 x$ and $k_2 x$, respectively. Now k_2 is essentially positive, so that there is always a downstream wake. On the other hand, we have $k_1 \gtrless 0$ according as $M^2 \lessgtr R R_M$. Thus, when μH_0^2 is less than $\varrho_0 U^2$, the second wake is also downstream, but when it is greater the wake is upstream.

4. Boundary Conditions at the Sphere

Since, for simplicity, we assume that the permeability of the sphere is the same as that of the fluid, the magnetic intensity $\boldsymbol{h}$ must be continuous at the surface of the sphere. On the other hand, the disturbance velocity $\boldsymbol{v}$ of the fluid assumes the value $-\boldsymbol{i}$ there. From equations (9) and (10) we see that this places the conditions

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial r} - \frac{1}{2k_1}\frac{\partial\chi_1}{\partial r} + \frac{1}{2k_2}\frac{\partial\chi_2}{\partial r} + (\chi_1 - \chi_2)\cos\vartheta = \frac{\partial\psi}{\partial r}, \\
&\frac{\partial\varphi}{\partial\vartheta} - \frac{1}{2k_1}\frac{\partial\chi_1}{\partial\vartheta} + \frac{1}{2k_2}\frac{\partial\chi_2}{\partial\vartheta} - (\chi_1 - \chi_2)\sin\vartheta = \frac{\partial\psi}{\partial\vartheta},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial r} + \frac{\alpha_2}{2k_1}\frac{\partial\chi_1}{\partial r} - \frac{\alpha_1}{2k_2}\frac{\partial\chi_2}{\partial r} - (\alpha_2\chi_1 - \alpha_1\chi_2)\cos\vartheta = -\cos\vartheta, \\
&\frac{\partial\varphi}{\partial\vartheta} + \frac{\alpha_2}{2k_1}\frac{\partial\chi_1}{\partial\vartheta} - \frac{\alpha_1}{2k_2}\frac{\partial\chi_2}{\partial\vartheta} + (\alpha_2\chi_1 - \alpha_1\chi_2)\sin\vartheta = \sin\vartheta,
\end{aligned}
$$

on the functions φ, χ_1, χ_2, ψ at the sphere.

Formulas given by GOLDSTEIN (1929) for the purely viscous case enable us to obtain the solution of the present problem to any desired accuracy. However, we are only interested in the first-order effects in M, R, and R_M. For this purpose it is better to adopt the *ad hoc* method used by LAMB (1911) for the purely viscous case and later by CHESTER (1957) for the present problem. Its justification lies in the fact that, as far as it goes, it is equivalent to GOLDSTEIN's method.

Now the general solutions of (6) and (8) outside the sphere are, in the case of axial symmetry,

$$\varphi = \sum_{n=0}^{\infty} A_n \frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right), \quad \chi_1 = e^{k_1 x}\sum_{n=0}^{\infty} B_n \frac{\partial^n}{\partial x^n}\left(\frac{e^{\mp k_1 r}}{r}\right), \quad \chi_2 = e^{k_2 x}\sum_{n=0}^{\infty} C_n \frac{\partial^n}{\partial x^n}\left(\frac{e^{-k_2 r}}{r}\right)$$

where the $\mp$ sign in χ_1 corresponds to $k_1 \gtrless 0$ (the solution must vanish at infinity). The general solution of (11) inside the sphere is

$$\psi = \sum_{n=1}^{\infty} D_n r^{2n+1} \frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right),$$

for the case of axial symmetry. When these are substituted into the four boundary conditions and all functions are expanded in powers of k_1 and k_2, and then rearranged into spherical harmonics, we find that A_n, B_n, C_n, and D_n must satisfy the equations

$$-A_0 S_0 - 2A_1 S_1 + \cdots + B_0\left[\frac{1}{2k_1} S_0 + \left(-1 \pm \frac{1}{2} k_1\right) S_1 + \frac{1}{4} k_1 S_2 + O(k_1^2)\right] + \cdots$$
$$- C_0\left[\frac{1}{2k_2} S_0 + \left(-1 + \frac{1}{2} k_2\right) S_1 + \frac{1}{4} k_2 S_2 + O(k_2^2)\right] - \cdots = D_1 S_1 + \cdots,$$

$$A_1 S_0 + \cdots + B_0\left[\left(-\frac{1}{2} \pm \frac{1}{2} k_1\right) S_0 + \frac{1}{2} k_1 S_1 + O(k_1^2)\right] + \cdots$$
$$- C_0\left[\left(-\frac{1}{2} + \frac{1}{2} k_2\right) S_0 + \frac{1}{2} k_2 S_1 + O(k_2^2)\right] - \cdots = D_1 S_0 + \cdots,$$

$$-A_0 S_0 - 2A_1 S_1 + \cdots - \alpha_2 B_0\left[\frac{1}{2k_1} S_0 + \left(-1 \pm \frac{1}{2} k_1\right) S_1 + \frac{1}{4} k_1 S_2 + O(k_1^2)\right] - \cdots$$
$$+ \alpha_1 C_0\left[\frac{1}{2k_2} S_0 + \left(-1 + \frac{1}{2} k_2\right) S_1 + \frac{1}{4} k_2 S_2 + O(k_2^2)\right] + \cdots = S_1,$$

$$A_1 S_0 + \cdots - \alpha_2 B_0\left[\left(-\frac{1}{2} \pm \frac{1}{2} k_1\right) S_0 + \frac{1}{2} k_1 S_1 + O(k_1^2)\right] - \cdots$$
$$+ \alpha_1 C_0\left[\left(-\frac{1}{2} + \frac{1}{2} k_2\right) S_0 + \frac{1}{2} k_2 S_1 + O(k_2^2)\right] + \cdots = S_0.$$

Here S_0, S_1, S_2 are the spherical harmonics

$$1, \quad \left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right]_{r=1}, \quad \left[\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)\right]_{r=1},$$

and the dots stand for omitted terms in A_n, D_n $(n>1)$ and B_n, C_n $(n>0)$.

Since the spherical harmonics are independent, these provide an infinite set of equations for the coefficients A_n, B_n, C_n, and D_n. From this set the A_n and D_n can be eliminated, the first three of the resulting equations being

$$(1+\alpha_2) B_0\left[\frac{1}{2k_1} + O(k_1^2)\right] + \cdots - (1+\alpha_1) C_0\left[\frac{1}{2k_2} + O(k_2^2)\right] - \cdots = 0,$$

$$\alpha_2 B_0\left[-2 \pm \frac{3}{2} k_1 + O(k_1^2)\right] + \cdots - \alpha_1 C_0\left[-2 + \frac{3}{2} k_2 + O(k_2^2)\right] - \cdots = -3,$$

$$(1+\alpha_2) B_0\left[\frac{1}{2} + O(k_1^2)\right] + \cdots - (1+\alpha_1) C_0\left[\frac{1}{2} + O(k_2^2)\right] - \cdots = 0.$$

Sufficient accuracy is obtained by solving the first two of these equations with $B_n = C_n = 0$ for $n>0$:

$$B_0 = C_0 = \frac{3}{2(\alpha_2 - \alpha_1)}\left[1 + \frac{3(\pm\alpha_2 k_1 - \alpha_1 k_2)}{4(\alpha_2 - \alpha_1)}\right].$$

The bracket is correct to the first order in k_1 and k_2. The corresponding value of A_0, which, as we shall show in the next section, determines the drag, is

(12)
$$\begin{aligned} A_0 &= -\frac{3(k_1-k_2)}{4(\alpha_2-\alpha_1)k_1k_2}\left[1+\frac{3(\pm\alpha_2 k_1-\alpha_1 k_2)}{4(\alpha_2-\alpha_1)}\right] \\ &= -\frac{3R_M}{2(RR_M-M^2)}\left[1+\frac{3}{8}K\right], \end{aligned}$$

where
$$K=\begin{cases} R & \text{for} \quad k_1>0, \\ \dfrac{2M^2+R^2-RR_M}{\sqrt{(R-R_M)^2+4M^2}} & \text{for} \quad k_1<0. \end{cases}$$

5. The Drag on the Sphere

The drag D due to the pressure and viscous forces on the sphere is $\varrho_0 U^2 a^2/R$ times

(13)
$$-\int [l p + m\zeta - n\eta]\, dS,$$

[*cf.* LAMB (1932)] where (l, m, n) are the direction-cosines of the outward-drawn normal to the sphere S: $r=1$, while η and ζ are the y- and z-components of curl $\boldsymbol{v}$. The drag due to the Maxwell stresses is zero: it is proportional to the flux of $\boldsymbol{h}$ through the surface of the sphere.

Now, according to equations (9), the integrand in (13) can be written in the form
$$l\left[-R\frac{\partial\varphi}{\partial x}+\frac{M^2}{R_M}\left\{\left(\frac{1}{2k_1}\frac{\partial\chi_1}{\partial x}-\frac{1}{2k_2}\frac{\partial\chi_2}{\partial x}\right)-(\chi_1-\chi_2)\right\}\right]+$$
$$+m\left[\alpha_2\frac{\partial\chi_1}{\partial y}-\alpha_1\frac{\partial\chi_2}{\partial y}\right]+n\left[\alpha_2\frac{\partial\chi_1}{\partial z}-\alpha_1\frac{\partial\chi_2}{\partial z}\right],$$

while on the sphere equations (7) show that
$$\frac{1}{2k}\operatorname{grad}\chi-\chi\,\boldsymbol{i}=\frac{1}{\alpha_1-\alpha_2}\left[R_M\boldsymbol{i}-\alpha\boldsymbol{h}+(\alpha+R_M)\operatorname{grad}\varphi\right]$$

for $k=k_1$, k_2 *etc.* Hence the integrand is equal to
$$-\left(R-\frac{M^2}{R_M}\right)\frac{\partial\varphi}{\partial r}-\frac{M^2}{R_M}h_r,$$

where h_r is the component of $\boldsymbol{h}$ along the outward normal. Clearly the h_r-term makes no contribution to the drag, and
$$\frac{D}{D_S}=\frac{D}{6\pi\varrho_0 U^2 a^2/R}=\frac{R}{6\pi}\left(1-\frac{M^2}{RR_M}\right)\int\frac{\partial\varphi}{\partial r}dS=-\frac{2R}{3}\left(1-\frac{M^2}{RR_M}\right)A_0,$$

where $D_S=6\pi\varrho_0\nu a U$ is the Stokes drag. Inserting the approximation (12) for A_0, this becomes
$$\frac{D}{D_S}=1+\frac{3}{8}\varkappa R,$$

where
$$\varkappa=\begin{cases} 1 & \text{for} \quad M^2<RR_M, \\ \sqrt{\dfrac{R_M}{R}}\left[\dfrac{2M^2/RR_M+R/R_M-1}{\sqrt{(\sqrt{R/R_M}-\sqrt{R_M/R})^2+4M^2/RR_M}}\right] & \text{for} \quad M^2>RR_M. \end{cases}$$

Thus, when the magnetic pressure μH_0^2 is less than the dynamic pressure $\varrho_0 U^2$ (*i.e.* $M^2/RR_M<1$), the magnetic field has no influence on the drag. The effect when $\mu H_0^2>\varrho_0 U^2$ is illustrated in the figure, where $\varkappa$ has been plotted against M^2/RR_M for various values of $R_M/R=\mu\sigma\nu$.

In the purely viscous case, $\sigma=0$, $M=R_M=0$, and $\varkappa=1$, so that we recapture OSEEN's result. On the other hand, when R and R_M are negligible, $\varkappa=M/R$ and we obtain the first two terms of CHESTER's (1957) formula (1). In the figure this corresponds to the asymptotic behavior of the curves for M^2/RR_M large.

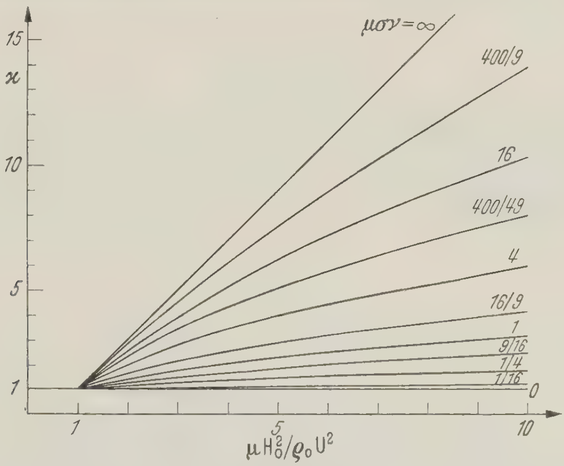

Fig. 1. Variation of the drag factor $\varkappa$ with $\mu H_0^2/\varrho_0 U^2$ for various values of $\mu\,\sigma\,\nu$

Note added in proof. As was stated in the introduction to the paper, Dr. R. VAN BLERKOM obtained similar results in his Ph. D. thesis at Harvard University. This was written under Professor G. F. CARRIER and is dated May 1959, some two months prior to my starting work in this area. In the same month, Professor CARRIER very briefly presented the results at a meeting I attended.

I had been made aware of these facts just before submitting the paper to this journal. At that time it appeared that Dr. VAN BLERKOM's interests were being safeguarded by his concurrent publication in another journal; there was no doubt in my mind that Professor CARRIER's announcement had made no impression on me. However, since VAN BLERKOM's publication has been delayed, it now seems appropriate to make the situation clear.

Research sponsored by the Office of Ordnance Research, U.S.Army, under Contract DA-36-034-ORD-1486.

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(Received October 26, 1959)

The Method of Extremal Points and Dirichlet's Problem in the Space of Two Complex Variables

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Communicated by S. BERGMAN

Introduction

NEVANLINNA's theory yields information about the distribution of values of a meromorphic function of one complex variable. In an attempt to generalize this theory to functions of two or more complex variables we meet two essential difficulties: 1° the zeros of analytic functions of two complex variables lie on segments of analytic surfaces, not at isolated points, and 2° the formulation and the solution of a boundary value problem analogous to the Dirichlet problem is not immediate. In contrast to the case of one variable there does not exist in general a pluriharmonic function (*i.e.* the real part of an analytic function of two complex variables) which assumes prescribed values on the three-dimensional boundary of the given domain. In connection with the second problem S. BERGMAN classifies the domains in the space of two complex variables (see [*1*]) into 1° domains such that to every point P of the boundary there exists an analytic function which assumes at P a value greater than that at any other point (*e.g.* the hypersphere), and 2° domains where only a certain part of the boundary has this property; it is called the *maximum boundary manifold* and is denoted in the literature as the *Bergman-Šilov boundary*. Further, BERGMAN showed that there exists a class of domains for which the maximum boundary manifold F is two-dimensional[1] and to every real continuous function given on F and satisfying certain additional conditions there exists a pluriharmonic function which is uniquely determined in the four-dimensional domain considered and which assumes on F the prescribed values (see [*2*]).

Since not every given boundary function satisfies the above mentioned additional conditions, BERGMAN introduces a larger class of functions (functions of extended class, see [*3*]) than the pluriharmonic functions; use of this class enables solution of the boundary problem with values prescribed on the Bergman-Šilov boundary. Introduction of the functions of extended class can be effected in various ways. In connection with these investigations D. B. LOWDENSLAGER [*19*] and H. J. BREMERMANN [*4*] defined different extended classes of functions.

[1] In this case F is called the distinguished boundary surface.

The aim of the present paper is to show how we can obtain a certain extended class of functions using the method of extremal points introduced and developed by F. LEJA (see [*13*], [*14*], [*15*]) and others ([*7*], [*8*], [*9*]). The class obtained is a certain subclass of plurisubharmonic functions, and the solution of the boundary-value problem considered is expressed as a generalized potential of a simple layer (see p. 418).

In Section 1 we generalize the method of extremal points to the space of two complex variables. In Section 2 a specialization of functions of extended class is made. In this latter case the original proofs based on the use of Stieltjes integrals can be replaced by an elementary method. An example illustrating this approach is given.

It should be mentioned that our results can be generalized to the case of analytic functions of n complex variables.

1. The method of extremal points in the space of two complex variables

As indicated in the introduction we use the method of extremal points in order to construct the extended class of functions. First we introduce certain notions; later they will be specialized to the case in which we are interested.

1. Let E be a compact set in n-dimensional Euclidean space R_n and let $g(p, q)$ be a real, continuous function of two points p and q defined in R_n. Suppose that $g(p, q) = g(q, p) \geqq 0$ and $g(p, p) = 0$. Let $f(p)$ be a real, continuous function defined on E, $\lambda > 0$ a real parameter and $p^{(n)} = \{p_0^{(n)}, p_1^{(n)}, \ldots, p_n^{(n)}\}$ an arbitrary system of $n+1$ points of E. A system $q^{(n)}(\lambda) = \{q_0^{(n)}, q_1^{(n)}, \ldots, q_n^{(n)}\}$ of $n+1$ points of E is called an extremal system if the inequality

$$V_n(q^{(n)}(\lambda)) = \prod_{0 \leqq j < k \leqq n} g(q_j^{(n)}, q_k^{(n)}) \exp\{-\lambda[f(q_j^{(n)}) + f(q_k^{(n)})]\} \geqq V_n(p^{(n)})$$

holds for every system $p^{(n)} \in E$.

It is known (see [*13*]) that the limit

$$\lim_{n \to \infty} [V_n(q^{(n)}(\lambda))]^{\frac{2}{n(n+1)}} = d(E, g, \lambda) \geqq 0$$

exists, and we shall call it the *generalized capacity* of the set E.

Special cases. 1. When E is a plane set and $g(p, q) = |z - \zeta|$, where z and ζ are two points in the complex plane, then $d(E, |z-\zeta|, 0)$ is the transfinite diameter of the set E (see [*5*]).

2. In the case when $E = R_3$ and $g(p, q) = \exp\left\{-\frac{1}{pq}\right\}$ where pq denotes the Euclidean distance between two points p and q, $d\left(E, \exp\left\{-\frac{1}{pq}\right\}, 0\right)$ is equal to $\exp\left\{-\frac{1}{C(E)}\right\}$, where $C(E)$ is the Newtonian capacity of the set E (see [*21*]).

3. In the case when E is the space of two complex variables z, w and $g(p, q) = \frac{1}{2}|z\eta - w\zeta|$, $d(E, \frac{1}{2}|z\eta - w\zeta|, 0)$ is called the triangle capacity (l'écart de l'ensemble) of E (see [*16*]).

The first two kinds of capacity play an important role in the theory of conformal mapping and DIRICHLET's problem; the last one has been used to determine the domain of convergence of a series of homogeneous polynomials.

The inequality $d(E, g, \lambda) > 0$ depends on E and $g(p, q)$ but not on $f(p)$; if $d(E, g, 0) > 0$, then $d(E, g, \lambda)$ is positive. Suppose that the set E and the function g are such that $d(E, g, 0) > 0$.

To each system of extremal points $q^{(n)}(\lambda)$ there corresponds a function of the Borel set e defined by the relation

$$\mu_{n\lambda}^{(g)}(e) \equiv \mu_{n\lambda}(e) = \begin{cases} 0, & \text{if } e \text{ does not contain any point of } q^{(n)}(\lambda), \\ \dfrac{k}{n+1}, & \text{if } e \text{ contains } k \text{ points of } q^{(n)}(\lambda). \end{cases}$$

The functions $\mu_{n\lambda}(e)$ satisfy the inequalities $0 \leqq \mu_{n\lambda}(e) \leqq 1$ and therefore define a certain distribution of unit mass on E. Let μ_λ be the limit of a convergent subsequence chosen from $\{\mu_{n\lambda}\}$; then

$$\text{(1)} \quad \log d(E, g, \lambda) = \int_E \int_E \log g(p, q)\, d\mu_\lambda(p)\, d\mu_\lambda(q) - 2\lambda \int_E f(q)\, d\mu_\lambda(q) = I(\mu_\lambda) \geqq I(\tau),$$

where τ denotes an arbitrary distribution of unit mass on E $\Big($see [7], where the special case $g(p, q) = \exp\left\{-\dfrac{1}{pq}\right\}$ is considered$\Big)$.

Consider the function

$$v_\lambda(p) = \int_E \log g(p, q)\, d\mu_\lambda(q) - \lambda f(p)$$

defined on E, and let $E_\lambda(g) = E_\lambda$ be the support of the mass distribution $\mu_\lambda^{(g)} = \mu_\lambda$.

Theorem 1. *The function $v_\lambda(p)$ is equal to a constant almost everywhere*[1] *on E_λ* $\big($see [6]$\big)$.

Proof. Set[2]

$$\int_E v_\lambda(p)\, d\mu_\lambda(p) = \gamma_\lambda.$$

We cannot have $v_\lambda(p) \geqq \gamma_\lambda + \varepsilon$, $\varepsilon > 0$ at every point of E, and hence there exists a point $p_0 \in E_\lambda$ such that $v_\lambda(p_0) < \gamma_\lambda + \varepsilon$. The function $v_\lambda(p)$ is upper semicontinuous, and therefore this inequality holds in a neighborhood $O(p_0)$ of the point p_0. Suppose there exists a set $F \subset E$ such that $d(F, g, 0) > 0$ and $v_\lambda(p) \geqq \gamma_\lambda + 2\varepsilon$ on F. Set $\mu_\lambda[O(p_0)] = m$, and let $\sigma \equiv \sigma(e)$ be a function defined by the formulas

$$\begin{aligned} &\sigma(e) = -\mu_\lambda(e) && \text{on } O(p_0), \\ &\sigma(e) \geqq 0 && \text{on } F, \\ &\sigma(F) = m && \\ &\sigma(e) = 0 && \text{outside } O(p_0) + F \end{aligned}$$

and

$$A = \int_E \int_E \log g(p, q)\, d\sigma(p)\, d\sigma(q) > -\infty.$$

[1] Almost everywhere means except for a set $F \subset E_\lambda$ such that $d(F, g, 0) = 0$.
[2] $\gamma_\lambda = \log d(E, g, \lambda) + \lambda \int_E f(q)\, d\mu_\lambda(q)$.

The function $\nu=\mu_\lambda+h\sigma$, $0<h\leqq 1$ defines a new distribution of unit mass on E. From (1) we obtain

$$\begin{aligned}0 &\geqq \int_E\int_E \log g(p,q)\,d\nu(p)\,d\nu(q) - 2\lambda\int_E f(q)\,d\nu(q) - \\ &\quad - \int_E\int_E \log g(p,q)\,d\mu_\lambda(p)\,d\mu_\lambda(q) + 2\lambda\int_E f(q)\,d\mu_\lambda(q) \\ &= h^2A + 2h\int_E v_\lambda(q)\,d\sigma(q) \\ &\geqq h^2A + 2h[-m(\gamma_\lambda+\varepsilon)+m(\gamma_\lambda+2\varepsilon)] \\ &= h^2A + 2mh\varepsilon.\end{aligned}$$

For sufficiently small $h>0$ the right side is >0, in contradiction to the inequality on the left side.

Therefore $v_\lambda(p)\leqq\gamma_\lambda$ almost everywhere on E (*cf.* footnote 1). We cannot have $v_\lambda(p_0)<\gamma_\lambda$ at a point $p_0\in E_\lambda$, since because of the upper semicontinuity of $v_\lambda(p)$ this inequality would hold in a neighborhood of p_0 and $\int_E v_\lambda(p)\,d\mu_\lambda(p)$ would be $<\gamma_\lambda$. Hence $v_\lambda(p)=\gamma_\lambda$ almost everywhere on E_λ. Q.E.D.

Let $u_\lambda(p)$ be defined as follows:

$$u_\lambda(p) = \frac{1}{\lambda}\int_E \log g(p,q)\,d\mu_\lambda(q).$$

From the upper semicontinuity of $u_\lambda(p)$ it follows that

$$\text{(2)}\qquad \overline{\lim_{p\to p_0\in E_\lambda}}\, u_\lambda(p)\leqq u_\lambda(p_0)\qquad \left(=\frac{\gamma_\lambda}{\lambda}+f(p_0)\quad\text{almost everywhere on } E_\lambda\right).$$

Suppose $g(p,q)\neq 0$ for $p\neq q$ and $\log[1/g(p,q)]$ satisfies the maximum principle of O. FROSTMAN, *i.e.* that for every distribution σ of positive mass on E, from the inequality

$$\frac{1}{\lambda}\int_E \log\frac{1}{g(p,q)}\,d\sigma(q)\leqq K<\infty$$

on the support E_σ results the same inequality in the whole space. Then

$$\text{(3)}\qquad \underline{\lim_{p\to p_0\in E_\lambda}}\,\frac{1}{\lambda}\int_E \log g(p,q)\,d\mu_\lambda(q)\geqq\frac{\gamma_\lambda}{\lambda}+f(p_0).$$

For let p_0 be a point of E_λ and s, S be two spheres centered at p_0 such that

$$0>\frac{1}{\lambda}\int_S \log g(p_0,q)\,d\mu_\lambda(q)>-\varepsilon$$

and

$$\left|\frac{1}{\lambda}\int_{E-S}\log g(p,q)\,d\mu_\lambda(q)-\frac{1}{\lambda}\int_{E-S}\log g(p',q)\,d\mu_\lambda(q)\right|<\varepsilon$$

for p, $p'\in s$. Therefore we have the following inequalities:

$$\text{(i)}\qquad \frac{1}{\lambda}\int_s \log g(p,q)\,d\mu_\lambda(q)\geqq\inf_{p\in E_\lambda s}\left\{\frac{1}{\lambda}\int_s \log g(p,q)\,d\mu_\lambda(q)\right\}>$$

$$>\frac{1}{\lambda}\int_s \log g(p_1,q)\,d\mu_\lambda(q)-\varepsilon,$$

where p_1 is a suitably chosen point on $E_\lambda s$;

(ii)
$$\lim_{p\to p_0} \frac{1}{\lambda} \int_{S-s} \log g(p,q)\, d\mu_\lambda(q) = \frac{1}{\lambda} \int_{S-s} \log g(p_0,q)\, d\mu_\lambda(q) > $$
$$> \frac{1}{\lambda} \int_S \log g(p_0,q)\, d\mu_\lambda(q) > -\varepsilon,$$

(iii)
$$\frac{1}{\lambda} \int_{E-S} \log g(p,q)\, d\mu_\lambda(q) > \frac{1}{\lambda} \int_{E-S} \log g(p_1,q)\, d\mu_\lambda(q) - \varepsilon, \qquad p \in s.$$

(i), (ii) and (iii) yield

$$\varliminf_{p\to p_0 \in E_\lambda} \frac{1}{\lambda} \int_E \log g(p,q)\, d\mu_\lambda(q) \geqq \frac{1}{\lambda} \int_E \log g(p_1,q)\, d\mu_\lambda(q) - 3\varepsilon.$$

Therefore

$$\varliminf_{p\to p_0 \in E_\lambda} \frac{1}{\lambda} \int_E \log g(p,q)\, d\mu_\lambda(q) \geqq \lim_{\substack{p_1\to p_0\\ p_1\in E_\lambda}} \frac{1}{\lambda} \int_E \log g(p_1,q)\, d\mu_\lambda(q) = \frac{\gamma_\lambda}{\lambda} + f(p_0)$$

almost everywhere on E_λ.

From (2) and (3) we deduce the existence of the limit

$$\lim_{p\to p_0 \in E_\lambda} \frac{1}{\lambda} \int_E \log g(p,q)\, d\mu_\lambda(q) = \frac{\gamma_\lambda}{\lambda} + f(p_0)$$

almost everywhere on E_λ.

When the function $\log[1/g(p,q)]$ satisfies the principle of energy (see [*6*], [*20*], [*22*]), that is if

$$\int_E\int_E \log \frac{1}{g(p,q)}\, d\tau(p)\, d\tau(q), \qquad \tau(E) = 0,$$

is always $\geqq 0$ and is equal to 0 only when $\tau = 0$, then 1° the function μ_λ is unique (and therefore the sequence $\{\mu_{n\lambda}\}$ is convergent), 2° $\mu_\lambda \to \mu_0$ when $\lambda \to 0$ (see [*12*]) and therefore $E_\lambda \to E_0$.

1°. In fact, if two different functions σ_λ and μ_λ realised the upper bound of the expression

$$I(\tau) \equiv \int_E\int_E \log g(p,q)\, d\tau(p)\, d\tau(q) - 2\lambda \int_E f(q)\, d\tau(q),$$

then

$$0 = \int_E\int_E \log g(p,q)\, d\sigma_\lambda(p)\, d\sigma_\lambda(q) - 2\lambda \int_E f(q)\, d\sigma_\lambda(q) -$$
$$- \int_E\int_E \log g(p,q)\, d\mu_\lambda(p)\, d\mu_\lambda(q) + 2\lambda \int_E f(q)\, d\mu_\lambda(q).$$

Set $\sigma_\lambda = \mu_\lambda + \nu$. We obtain

$$0 = -2\lambda \int_E f(q)\, d\nu(q) + \int_E\int_E \log g(p,q)\, d\nu(p)\, d\nu(q) + 2\int_E\int_E \log g(p,q)\, d\mu_\lambda(p)\, d\nu(q)$$
$$= 2\int_E v_\lambda(q)\, d\nu(q) + \int_E\int_E \log g(p,q)\, d\nu(p)\, d\nu(q)$$
$$\leqq 2[\gamma_\lambda \nu(E_\lambda) + \gamma_\lambda \nu(E - E_\lambda)] + \int_E\int_E \log g(p,q)\, d\nu(p)\, d\nu(q),$$

and, since $\nu(E_\lambda)+\nu(E-E_\lambda)=0$, it follows that

$$0 \leqq \int_E \int_E \log g(p,q)\, d\nu(p)\, d\nu(q).$$

$\log[1/g(p,q)]$ satisfies the principle of energy, and therefore the right-hand side of this inequality is $\leqq 0$, the sign $=$ holding only in the case $\nu \equiv 0$. Therefore $\sigma_\lambda = \mu_\lambda$.

2°. Suppose $\lambda_n \to 0$. We have

$$\log d(E,g,\lambda_n) - \log d(E,g,0) = \int_E \int_E \log g(p,q)\, d\mu_{\lambda_n}(p)\, d\mu_{\lambda_n}(q) - \\ - 2\lambda_n \int_E f(q)\, d\mu_{\lambda_n}(q) - \int_E \int_E \log g(p,q)\, d\mu_0(p)\, d\mu_0(q).$$

Set $\sigma_n = \mu_{\lambda_n} - \mu_0$; then

$$\log d(E,g,\lambda_n) - \log d(E,g,0) = \int_E \int_E \log g(p,q)\, d\sigma_n\, d\sigma_n + \\ + 2\int_E \int_E \log g(p,q)\, d\mu_0\, d\sigma_n - 2\lambda_n \int_E f(q)\, d\mu_{\lambda_n}.$$

The left-hand side tends towards 0, when $n\to\infty$, because $d(E,g,\lambda_n)$ is continuous with respect to λ_n (see [*11*], [*12*]). On the other hand

$$2\int_E \int_E \log g(p,q)\, d\mu_0\, d\sigma_n \leqq 2 \log d(E,g,0)\cdot[\sigma_n(E_0)+\sigma_n(E-E_0)] = 0.$$

Therefore

$$0 \leqq \overline{\lim_{n\to\infty}} \int_E \int_E \log g(p,q)\, d\sigma_n\, d\sigma_n.$$

Let σ be the limit of a convergent subsequence chosen from σ_n; then

$$0 \leqq \int_E \int_E \log g(p,q)\, d\sigma\, d\sigma.$$

From the principle of energy it follows that equality can hold only when $\sigma \equiv 0$.

Remarks. A. For some special cases, *e.g.* $n=2$, $g(p,q)=|z-\zeta|$ or $n=3$, $g(p,q)=\exp\left\{-\frac{1}{pq}\right\}$, the continuity of the function $u_\lambda(p)$ has been proved when E is sufficiently regular (see [*15*], [*10*]). Similarly in these two cases it has been proved that under some additional conditions on the set E and the function $f(p)$ there exists a $\lambda_0 > 0$ such that $E_\lambda = E$ for $\lambda > 0$, $\lambda < \lambda_0$ (see [*9*], [*10*]).

B. Let E be the boundary of a bounded domain $D \subset R_n$, and let

$$\psi(u) \equiv \sum_{\alpha,\beta=0}^{n} a_{\alpha\beta}(p)\, u''_{x_\alpha x_\beta}(p) + \sum_{\alpha=1}^{n} b_\alpha(p)\, u'_{x_\alpha}(p) + c(p)\, u(p) = 0$$

be a linear differential equation of elliptic type. Suppose $\Gamma(p,q)$ is the fundamental solution of $\psi(u)=0$ which satisfies the following condition: $\Gamma(p,q)=\Gamma(q,p)>0$, $\frac{1}{\Gamma(p,p)}=0$. Set $g(p,q)=\exp\{-\Gamma(p,q)\}$ and suppose $\Gamma(p,q)$ satisfies the maximum principle of FROSTMAN. If $d(E,\exp\{-\Gamma(p,q)\},0)\neq 0$, then the corresponding function $u_\lambda(p)=\int_E \Gamma(p,q)\, d\mu_\lambda(q)$ is the solution of the equation $\psi(u)=0$ which assumes on E_λ the value $\frac{\gamma_\lambda}{\lambda}+f(p)$.

C. Let D be a four-dimensional bounded domain in the space of two complex variables z, w, and let $f(p)$ be a real continuous function defined on the boundary E of D. Set

$$g(p, q) = |h(p, q)|$$

where $h(p, q)$ is a continuous function of two points $p=(z, w)$, $q=(\zeta, \eta)$ defined in a domain[1] $G \supset D+E$ and satisfying the following conditions:

1. $|h(p, q)| = |h(q, p)| \geqq 0$, $h(p, p) = 0$,
2. $h(p, q)$ is an analytic function of $p=(z, w)$ for fixed $q=(\zeta, \eta)$.

Suppose that $d(E, |h|, 0)$ is positive. Using the method of extremal points indicated above, we obtain the function

$$u_\lambda(p) = \frac{1}{\lambda} \int\limits_E \log|h(p, q)|\, d\mu_\lambda(q)$$

which is plurisubharmonic[2] at every finite point. If $h(p, q) \neq 0$ for $p \in D$ and $q \in E$, then $u_\lambda(p)$ is a pluriharmonic function in D. We have

$$u_\lambda(p) \begin{cases} \leqq \dfrac{\gamma_\lambda}{\lambda} + f(p) & \text{almost everywhere on } E, \\[2ex] = \dfrac{\gamma_\lambda}{\lambda} + f(p) & \text{almost everywhere on } E_\lambda. \end{cases}$$

If the function $\log \dfrac{1}{|h(p, q)|}$ satisfies the principle of energy, then $\mu_\lambda \to \mu_0$ for $\lambda \to 0$ and therefore $E_\lambda \to E_0$ (see [12]).

From the upper semicontinuity of the function $u_\lambda(p)$ follows

$$(*) \qquad \overline{\lim_{p \to p_0 \in E_\lambda}}\, u_\lambda(p) \leqq u_\lambda(p_0).$$

We shall prove that *for every "regular" point $p_0 \in E_\lambda$ the limit*

$$\lim_{p \to p_0 \in E_\lambda} u_\lambda(p) = u_\lambda(p_0)$$

exists.

Let p_0 be a point of E, and let $F(p_0) = E \cap \{h(p_0, q) = 0\}$ be the product of the sets E and $\{h(p_0, q) = 0\}$. From the continuity of the function $h(p, q)$ it follows that for $|p p_0| < r_1$ the set $E \cap \{h(p, q) = 0\}$ is contained in the set $G_\delta(p_0) = E \cap F_\delta(p_0)$, where $F_\delta(p_0)$ denotes the hull of the set $F(p_0)$ with radius $\delta > 0$.

Let $\varepsilon > 0$ be an arbitrary number, and let p_0 be a point of E_λ. If δ is sufficiently small, then

$$\left| \frac{1}{\lambda} \int\limits_{G_\delta(p_0)} \log|h(p_0, q)|\, d\mu_\lambda(q) \right| < \varepsilon.$$

Therefore

$$(**) \quad \lim_{p \to p_0 \in E_\lambda} \frac{1}{\lambda} \int\limits_{E - G_\delta(p_0)} \log|h(p, q)|\, d\mu_\lambda(q) = \frac{1}{\lambda} \int\limits_{E - G_\delta(p_0)} \log|h(p_0, q)|\, d\mu_\lambda(q) > u_\lambda(p_0) - \varepsilon.$$

[1] G can be the whole space.

[2] A real function $V(p)$ is plurisubharmonic in a domain if the following conditions are satisfied: (a) $-\infty \leqq V(p) < \infty$, (b) $V(p)$ is upper semicontinuous, (c) the restriction of $V(p)$ to any analytic plane P of complex dimension one is subharmonic in the intersection PD.

We shall call a point $p_0 \in E_\lambda$ "regular" if for every $\varepsilon > 0$ there exist two numbers $\delta_\varepsilon(p_0) > 0$ and $r(\varepsilon) > 0$ such that

$$(***) \qquad \left| \frac{1}{\lambda} \int_{G\,\delta_\varepsilon(p_0)} \log|h(p,q)|\, d\mu_\lambda(q) \right| < \varepsilon \quad \text{for} \quad |p\,p_0| < r(\varepsilon).$$

If $p_0 \in E_\lambda$ is regular, then from (**) and (***) follows

$$\underline{\lim}_{p \to p_0 \in E_\lambda} u_\lambda(p) \geqq u_\lambda(p_0),$$

and therefore (see (*)) the limit

$$\lim_{p \to p_0 \in E_\lambda} u_\lambda(p) = u_\lambda(p_0)$$

exists. Q.E.D.

Suppose $p_0 \in E_\lambda$ is a regular point with respect to the Dirichlet problem for the domain D, and suppose there exists a cone c contained in D with the vertex at p_0. *We shall show that* $u_\lambda(p_0) = \frac{\gamma_\lambda}{\lambda} + f(p_0)$. Let s be a sphere with radius r centered at p_0. We denote by $u'_\lambda(p)$ and $u''_\lambda(p)$ the following functions:

$$u'_\lambda(p) = \frac{1}{\lambda} \int_{G_\delta(p_0)} \log|h(p,q)|\, d\mu_\lambda(q), \qquad u''_\lambda(p) = \frac{1}{\lambda} \int_{E - G_\delta(p_0)} \log|h(p,q)|\, d\mu_\lambda(q).$$

Let k be equal to the ratio $\dfrac{\text{volume of } cs}{\text{volume of } s}$. Suppose the number $\delta > 0$ is such that

$$\text{(a)} \qquad 0 > u'_\lambda(p_0) > -k\,\varepsilon, \qquad \varepsilon > 0.$$

We choose the radius r so small that

$$\text{(b)} \qquad u''_\lambda(p) > u''_\lambda(p_0) - \varepsilon \quad \text{and} \quad u'_\lambda(p) < 0 \quad \text{for} \quad p \in s.$$

Let $\omega(p)$ be the generalized solution of the Dirichlet problem for the domain D with the boundary value $f(p)$. The function $u_\lambda(p) - \omega(p)$ is subharmonic in D, $\leqq \gamma_\lambda/\lambda$ almost everywhere on E. Let m_{cs}, m'_{cs} and m''_{cs} be the average value of $u_\lambda(p) - \omega(p)$, $u''_\lambda(p)$ and $u'_\lambda(p)$ on cs respectively. We have $(\omega(p) < f(p_0) + \varepsilon$ in $cs)$

$$\frac{\gamma_\lambda}{\lambda} \geqq m_{cs} \geqq m'_{cs} + m''_{cs} - f(p_0) - \varepsilon.$$

From (a) and (b) follows

$$m'_{cs} = \frac{1}{k \text{ vol. } s} \int_{cs} u'_\lambda(p)\, d\sigma \geqq \frac{1}{k \text{ vol. } s} \int_{s} u'_\lambda(p)\, d\sigma \geqq \frac{1}{k}\, u'_\lambda(p_0) > -\varepsilon.$$

$$m''_{cs} \geqq u'_\lambda(p_0) - \varepsilon.$$

Therefore

$$\frac{\gamma_\lambda}{\lambda} \geqq u''_\lambda(p_0) - f(p_0) - 3\,\varepsilon \geqq u_\lambda(p_0) - f(p_0) - 3\,\varepsilon,$$

and since $\varepsilon > 0$ is arbitrarily small, we obtain $u_\lambda(p_0) \leqq \frac{\gamma_\lambda}{\lambda} + f(p_0)$.

On the other hand we have $u_\lambda(p_0) \geqq \frac{\gamma_\lambda}{\lambda} + f(p_0)$ for $p_0 \in E_\lambda$, and therefore

$$u_\lambda(p_0) = \frac{\gamma_\lambda}{\lambda} + f(p_0). \qquad \text{Q.E.D.}$$

From the definition of the extremal points and from the analyticity of the function $h(p, q)$ with respect to $p=(z, w)$, it follows that in the case $\lambda=0$ we can choose the extremal points on the Bergman-Šilov boundary B of the domain D. (In the case of a domain with the distinguished boundary surface F we have $F=B$.) We denote by E_0 the support of the mass distribution μ_0 defined by the extremal points in the case $\lambda=0$. In general $E_0 \neq B$. If $\log \frac{1}{|h(p,q)|}$ satisfies the principle of energy, we have $E_\lambda \to E_0$ for $\lambda \to 0$.

We set $\lambda=1$, and we consider the functions

$$u_1^{(h)}(p)-\gamma_1^{(h)} \equiv u_1(p)-\gamma_1,$$

which correspond to different functions $h(p, q)$ satisfying Conditions 1. and 2. (p. 418). For $p \in D$ we define the following function:

$$w_1(p)=\overline{\lim_{q\to p}}\left\{\sup_{(h)}\left[u_1^{(h)}(q)-\gamma_1^{(h)}\right]\right\}.$$

Since every function $u_1^{(h)}(q)$ is plurisubharmonic in D, the function $w(p)$ is plurisubharmonic in D (see [4]).

Suppose the domain D is regular at every point $p_0 \in E$ (see p. 419). For every function $h(p, q)$ we have $u_1^{(h)}(q)-\gamma_1^{(h)} \leqq f(q)$ for $q \in E$. If[1] $q_0 \in E_1^{(h)}$, then $u_1^{(h)}(q_0)-\gamma_1^{(h)}=f(q_0)$ and therefore $\sup_{(h)}[u_1^{(h)}(q_0)-\gamma_1^{(h)}]=f(q_0)$. This formula is valid for every point $q_0 \in \sum_{(h)} E_1^{(h)}$.

Let $\omega(p)$ be the solution of the Dirichlet problem for the domain D and the boundary values $f(p)$. We have $u_1^{(h)}(q)-\gamma_1^{(h)}-\omega(q) \leqq 0$ on E, and, since the function $u_1^{(h)}(q)-\gamma_1^{(h)}-\omega(q)$ is subharmonic in D, this inequality holds in D. Therefore $\sup_{(h)}[u_1^{(h)}(q)-\gamma_1^{(h)}-\omega(q)] \leqq 0$ for $q \in D$ and $w_1(p)-\omega(p) \leqq 0$ in D, and consequently $w(p) \leqq \omega(p)$, $p \in D$.

Let q_0 be a point of the set $\sum_h E_1^{(h)}$; then

$$\overline{\lim_{p\to q_0}}\, w_1(p) \leqq \lim_{p\to q_0} \omega(p)=f(q_0)$$

and, if q_0 is "regular",

$$\underline{\lim_{p\to q_0}}\, w_1(p)=\underline{\lim_{p\to q_0}}\left\{\overline{\lim_{q\to p}}\left\{\sup_{(h)}\left[u_1^{(h)}(q)-\gamma_1^{(h)}\right]\right\}\right\} \geqq f(q_0).$$

Therefore[2] for regular q_0

$$\lim_{p\to q_0 \in \sum_h E_1^{(h)}} w_1(p)=f(q_0).$$

---

[1] $E_1^{(h)}$ is the support of the mass distribution $\mu_1^{(h)}$.

[2] Analogously we can set $\lambda=-1$ and introduce the function

$$w_{-1}(p)=\underline{\lim_{q\to p}}\left\{\inf_{(h)}\left[u_{-1}^{(h)}(q)-\gamma_{-1}^{(h)}\right]\right\}$$

which is plurisuperharmonic in D. We have

$$w_{-1}(p) \geqq \omega(p) \quad \text{and} \quad \lim_{p\to q_0} w_{-1}(p)=f(q_0), \quad \text{for a "regular" point} \quad q_0 \in \sum_h E_{-1}^{(h)}.$$

Remark. Let $D=D_1\times D_2$ be a product domain, where D_1 and D_2 are simply connected domains lying on the z and w plane respectively. Suppose the boundary E_1 and E_2 of the domains D_1 and D_2 are regular Jordan curves such that the angle between the tangent and the real axis expressed as the function of the length satisfies the Lipschitz condition. Let $f(p)$ be equal to $f_1(z)+f_2(w)$, where $f_1(z)$ and $f_2(w)$ satisfy the Lipschitz condition, and let $h(p,q)=(z-\zeta)(w-\eta)$. Then for sufficiently small $\lambda>0$ the function $u_\lambda(p)$ is pluriharmonic in D, continuous on $D+E$, $E=E_1\times E_2$ and equal to $f(p)$ at every point $p\in E$ (see [*10*]).

2. Specialization of the function $h(p, q)$

The considerations in this section give an explicit construction of a function of the extended class (a plurisubharmonic function).

Let D be a four dimensional bounded domain in the space C_2 of two complex variables z, w and $f(z,w)$ a real, continuous function defined on the boundary E of D. Set $h(p,q)=z\eta-w\zeta$, $p=(z,w)$, $q=(\zeta,\eta)$. It is known (see [*16*]) that $d(E,|z\eta-w\zeta|,0)$ is >0 if E does not contain the origin $(0,0)$ of the coordinate system and contains two points $p=(z,w)$, $q=(\zeta,\eta)$ whose "triangle distance" $|pq|=|z\eta-w\zeta|$ is >0. We assume that the origin (0,0) lies outside of E.

We denote by $\Phi_n^{(j)}(p;p^{(n)})$, $j=0,1,\ldots,n$ the following functions:

$$\Phi_n^{(j)}(p;p^{(n)})=\left[\prod_{\substack{k=0\\k\neq j}}^{n}\frac{p\,p_k}{p_j\,p_k}\right]\exp[n\lambda f(p_j)] \tag{4}$$

where $p\,p_k=(zw_k-z_kw)$. $\Phi_n^{(j)}(p;p^{(n)})$ are homogeneous polynomials of the variables z and w.

For a fixed point $p\in C_2$ we set

$$\Phi_n(p;\lambda)=\inf_{p^{(n)}\in E}\left\{\max_{(j)}|\Phi_n^{(j)}(p;p^{(n)})|\right\},\qquad n=1,2,\ldots. \tag{5}$$

It can be proved (see [*16*], where the case $\lambda=0$ is considered) that the functions (5) satisfy the inequalities

$$\Phi_{\mu+\nu}(p;\lambda)\geqq\Phi_\mu(p;\lambda)\,\Phi_\nu(p;\lambda),\qquad \mu,\nu=1,2,\ldots$$

and, therefore (since $d(E,|z\eta-w\zeta|,0)>0$), the limit

$$\Phi(p;\lambda)=\lim_{n\to\infty}\sqrt[n]{\Phi_n(p;\lambda)} \tag{6}$$

exists at every finite point $p=(z,w)$ of C_2.

Consider the functions $\Phi_n^{(j)}(p;q^{(n)}(\lambda))$ where $q^{(n)}(\lambda)$ denotes the n^{th} extremal system of points with respect to the function $h(p,q)=z\eta-w\zeta$. From the definition of the extremal points $q^{(n)}(\lambda)$ it follows that

$$|\Phi_n^{(j)}(p;q^{(n)}(\lambda))|\leqq\exp[n\lambda f(p)],\qquad j=0,1,\ldots,n \tag{7}$$

on E. We set

$$\Delta^{(j)}(q^{(n)}(\lambda))=\prod_{\substack{k=0\\k\neq j}}^{n}(z_jw_k-z_kw_j)\exp\{-\lambda[f(z_j,w_j)+f(z_k,w_k)]\},\quad j=0,1,\ldots,n$$

and suppose that

$$|\Delta^{(0)}(q^{(n)}(\lambda))| \leqq |\Delta^{(j)}(q^{(n)}(\lambda))| \quad \text{for } j=0,1,\ldots,n.$$

At every point $(z, w) = p$ of C_2 which satisfies the condition

(8) $$\delta(p, E) = \inf_{(\zeta,\eta)\in E} |z\eta - w\zeta| > 0,$$

the limit

(9) $$\lim_{n\to\infty} \sqrt[n]{|\Phi_n^{(0)}(p; q^{(n)}(\lambda))|} = \Phi(p; \lambda)$$

exists. On the other hand at every point $p \in C_2$ the sequence

$$\left\{\left[\sum_{j=0}^{n} |\Phi_n^{(j)}(p; q^{(n)}(\lambda))|\right]^{\frac{1}{n}}\right\} = \{\omega_n(p; \lambda)\}$$

is convergent to $\Phi(p; \lambda)$ (see [*16*]):

(10) $$\lim_{n\to\infty} \omega_n(p; \lambda) = \Phi(p; \lambda).$$

The function $\Phi(p; \lambda)$ has the following properties[1]:

1. For every point $p \neq 0$, $\Phi(p; \lambda) > 0$ and $\Phi(p; \lambda)$ satisfy the inequalities

$$\frac{|z| + |w|}{2A} e^{\lambda m} \leqq \Phi(p; \lambda) \leqq \frac{A(|z| + |w|)}{2d(E, |z\eta - w\zeta|, 0)} e^{\lambda M},$$

where $m = \inf_{p\in E} f(p)$, $M = \sup_{p\in E} f(p)$ and $A = \sup_{p\in E} (|z|, |w|)$.

2. $\Phi(p; \lambda) = 0$ at $p = 0$ and is continuous there.

3. For an arbitrary complex number c

$$\Phi(cp; \lambda) = |c| \, \Phi(p; \lambda).$$

4. Let S be a set of points of C_2 which have positive triangle distance (8) from the set E. S is an open set because $\delta(p, E)$ is continuous function of the point p. The convergence (9) is uniform in every closed set $G \subset S$, and therefore $\Phi(p; \lambda)$ is equal in S to the absolute value of an analytic function.

5. The function $\Phi(p; \lambda)$ is lower semicontinuous everywhere.

6. On the set E

(11) $$\Phi(p; \lambda) \leqq \exp[\lambda f(p)].$$

7. When $\exp f(p) = \sqrt[r]{|Q_k(p)|}$ where $Q_k(p)$ is an arbitrary homogeneous polynomial of degree k, then

$$\Phi\left(p; \frac{r}{k}\right) = \exp\left[\frac{r}{k} f(p)\right] \quad \text{on } E.$$

8. Let E be contained in the set of points defined by $|Q_k(p)| = c > 0$. Then

$$\Phi(p; \lambda) \geqq \exp[\lambda m] \quad \text{on } E.$$

9. The function $\Phi(p; \lambda)$ satisfies everywhere the inequalities

$$\Phi(p; 0) \exp[\lambda m] \leqq \Phi(p; \lambda) \leqq \Phi(p; 0) \exp[\lambda M].$$

---

[1] Properties 1—9 can be easily proved; *cf.* [*16*]. The main tool is the Lagrange interpolation formula.

Remark. Using the earlier representation of Section 1, we can write

$$\log \Phi(p;0) = \int_E \log|z\eta - w\zeta|\, d\mu_0(\zeta,\eta) - \log d(E, |z\eta - w\zeta|, 0),$$
$$\log \Phi(p;\lambda) = \int_E \log|z\eta - w\zeta|\, d\mu_\lambda(\zeta,\eta) - \gamma_\lambda = u_\lambda(p) - \gamma_\lambda.$$

Let E_λ^* be the set of points of accumulation of the set of all extremal point systems

$$\begin{array}{lll} q_0^{(1)}, & q_1^{(1)}, & \\ q_0^{(2)}, & q_1^{(2)}, & q_2^{(2)} \\ \ldots & \ldots & \ldots, \end{array}$$

and suppose that every point $p_0 = (z_0, w_0) \in E$ possesses the following property:

(H) *for every sphere $K(p_0, r)$, the set $E \cap K(p_0, r)$ contains an arc γ of a curve $C \subset E$, $p_0 \in \gamma$ whose triangle capacity $d(\gamma, |z\eta - w\zeta|, 0)$ is positive.*

Theorem 2. *Let E satisfy the condition (H). Then for every point $p_0 \in E_\lambda^*$ the limit*

$$\lim_{p \to p_0 \in E_\lambda^*} \Phi(p;\lambda) = \exp[\lambda f(p_0)] \tag{12}$$

exists.

Proof. 1. Since $f(p)$ is continuous, there exists a sphere $K(p_0, r)$ such that $f(p) > f(p_0) - \varepsilon$, $\varepsilon > 0$, for $p \in E \cap K(p_0, r)$. Since $p_0 \in E_\lambda^*$, there exists an extremal point $q_j = q_j^{(n)}$ which belongs to the set $E \cap K(p_0, r)$. Therefore

$$|\Phi_n^{(j)}(p_0; q^{(n)}(\lambda))| \geqq \prod_{\substack{k=0\\k \neq j}}^{n} \left|\frac{p\, q_k}{q_j\, q_k}\right| \exp\{n\lambda[f(p_0) - \varepsilon]\},$$

and

$$\sum_{j=0}^{n} |\Phi_n^{(j)}(p_0; q^{(n)}(\lambda))| \geqq M(p_0; r, q^{(n)}(\lambda)) \exp\{n\lambda[f(p_0) - \varepsilon]\},$$

where $M(p_0; r, q^{(n)}(\lambda))$ is the maximum of the absolute values of those polynomials in the collection

$$\prod_{\substack{k=0\\k \neq j}}^{n} \left|\frac{p\, q_k}{q_j\, q_k}\right|, \qquad j = 0, 1, \ldots, n$$

for which the point q_j belongs to $E \cap K(p_0, r)$.

Since [1]

$$\overline{\lim_{n\to\infty}} \sqrt[n]{M(p_0; r, q^{(n)}(\lambda))} \geqq 1 - \sigma(r), \qquad \sigma(r) \to 0 \quad \text{when} \quad r \to 0,$$

we have

$$\Phi(p_0;\lambda) \geqq \exp\{\lambda[f(p_0) - \varepsilon]\} \cdot (1 - \sigma(r)). \tag{13}$$

The function $\Phi(p;\lambda)$ is lower semicontinuous; hence

$$\underline{\lim}_{p \to p_0 \in E_\lambda^*} \Phi(p;\lambda) \geqq \exp\{\lambda f(p_0)\}. \tag{14}$$

[1] See [*17*], where the proof in the case of one variable is given.

From (13) and (11) we obtain

(15) $$\Phi(p_0;\lambda) = \exp\{\lambda f(p_0)\}, \qquad p_0 \in E_\lambda^*.$$

2. Let $K(p_0, r)$ be a sphere such that $f(p) < f(p_0) + \eta$ for $p \in E \frown K(p_0, r)$. From (7) results the inequality

$$|\Phi_n^{(j)}(p; q^{(n)}(\lambda))| \leqq \exp\{n\lambda[f(p_0) + \eta]\}, \qquad p \in E \frown K(p_0; r).$$

The absolute values of the polynomials

$$\Phi_n^{(j)}(p; q^{(n)}(\lambda)) \exp\{-n\lambda[f(p_0) + \eta]\}$$

are therefore $\leqq 1$ on $E \frown K(p_0, r)$. From the condition (H) and from the theorem of C. LOSTER (see [18]) it follows that there exists a number N and a neighborhood Δ of the point p_0 such that for $n \geqq N$ and $p \in \Delta$

$$|\Phi_n^{(j)}(p; q^{(n)}(\lambda))| \exp\{-n\lambda[f(p_0) + \eta]\} < (1 + \varepsilon)^n, \qquad j = 0, 1, \ldots, n.$$

Therefore

$$\sum_{j=0}^{n} |\Phi_n^{(j)}(p; q^{(n)}(\lambda))| \leqq (1+n)(1+\varepsilon)^n \exp\{n\lambda[f(p_0) + \eta]\}, \qquad p \in \Delta.$$

From this last inequality it follows (see [10]) that

(16) $$\overline{\lim_{p \to p_0}} \Phi(p; \lambda) \leqq \exp\{\lambda f(p_0)\}.$$

(12) follows from (14) and (16). Q.E.D.

Remarks. 1. We denote by S_0 the set of points cp_0, where p_0 is an arbitrary point of E and c an arbitrary complex number. The function $\Phi(p; \lambda)$ is continuous in S_0 if it is continuous on E, because (see Property 3)

$$\lim_{p \to cp_0} \Phi(p; \lambda) = \lim_{p \to p_0} \Phi(cp; \lambda) = |c| \lim_{p \to p_0} \Phi(p; \lambda) = |c|\, \Phi(p_0; \lambda) = \Phi(cp_0; \lambda).$$

Since $C_2 = S_0 + S$, it follows that $\Phi(p; \lambda)$ is continuous at every finite point p when E satisfies the condition (H) and $\exp f(p) = |Q_k(p)|^{\frac{1}{r}}$. In fact, $\Phi(p; \lambda)$ is equal to $\exp\{\lambda f(p)\}$ on E (see Property 7), and from the lower semicontinuity of $\Phi(p; \lambda)$ it follows that

$$\underline{\lim_{p \to p_0 \in E}} \Phi(p; \lambda) \geqq \exp\{\lambda f(p_0)\}.$$

On the other hand since E satisfies the condition (H),

$$\overline{\lim_{p \to p_0 \in E}} \Phi(p; \lambda) \leqq \exp\{\lambda f(p_0)\}$$

(see the second part of the proof of Theorem 2).

2. Let $\widetilde{E}$ be a subset of E which contains all points p of E such that none of the points cp, $p \in E$, $c > 1$ belong to E. We shall call $\widetilde{E}$ the boundary of the set E with respect to the origin (0, 0). We have (see [16])

$$d(E, |z\eta - \zeta w|, 0) = d(\widetilde{E}, |z\eta - \zeta w|, 0),$$

and the function $\Phi(p; 0)$ constructed for the set E is identical with the corresponding function constructed for $\widetilde{E}$.

If E is contained in the set[1] $\{|Q_k(p)|=c>0\}$ where $Q_k(p)$ is an arbitrary homogeneous polynomial of degree k, and if E satisfies the condition (H), then $\Phi(p;0)$ is continuous at every finite point and $\Phi(p;0)=1$ for $p\in E$.

Theorem 3. *For every finite point $p \neq 0$ the limit*

$$\lim_{\lambda\to\lambda}\left[\frac{\Phi(\varrho;\lambda)}{\Phi(\varrho;0)}\right]^{\frac{1}{\lambda}}=\psi(p). \tag{17}$$

exists.

Proof. Let $0<\lambda_1<\lambda_2$ and

$$L_n^{(j)}(p;p^{(n)})=\prod_{\substack{k=0\\k\neq j}}^{n}\frac{p\,p_k}{p_j\,p_k},\qquad j=0,1,\ldots,n.$$

We denote by $p_\varepsilon^{(n)}(\lambda_1)$ a system of points of E such that

$$\varepsilon+\Phi_n(p;\lambda_1)>\max_{(j)}|\Phi_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|$$

for an arbitrary number $\varepsilon>0$. On the other hand we have

$$\Phi_n(p;\lambda_2)\leqq\max_{(j)}|\Phi_n^{(j)}(p;q^{(n)}(\lambda_2))|=|\Phi_n^{(s)}(p;q^{(n)}(\lambda_2))|=|L_n^{(s)}(p;q^{(n)}(\lambda_2))|\exp[n\,\lambda_2 f(q_s)],$$

where $q^{(n)}(\lambda_2)$ denotes the extremal system of points.

Using the interpolation formula of Lagrange we obtain

$$\begin{aligned}\Phi_n(p;\lambda_2)&\leqq\sum_{j=0}^{n}|L_n^{(j)}(p;p_\varepsilon^{n}(\lambda_1))|\,|L_n^{(s)}(p_j;q^{(n)}(\lambda_2))|\exp[n\,\lambda_2 f(q_s)]\\&\leqq\sum_{j=0}^{n}\exp[n\,\lambda_2 f(p_j)]\,|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|=\sum_{j=0}^{n}|\Phi_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|^{\frac{\lambda_2}{\lambda_1}}|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|^{1-\frac{\lambda_2}{\lambda_1}}\\&\leqq(n+1)\max_{(j)}|\Phi_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|^{\frac{\lambda_2}{\lambda_1}}\max_{(j)}|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|^{1-\frac{\lambda_2}{\lambda_1}}\\&<[\Phi_n(p;\lambda_1)+\varepsilon]^{\frac{\lambda_2}{\lambda_1}}(n+1)\max_{(j)}|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|^{1-\frac{\lambda_2}{\lambda_1}}.\end{aligned}\tag{18}$$

Let $q^{(n)}(0)$ be the n^{th} extremal system of points corresponding to $\lambda=0$; then

$$\begin{aligned}\Phi_n(p;0)&\leqq\max_{(j)}|L_n^{(j)}(p;q^{(n)}(0))|=|L_n^{(k)}(p;q^{(n)}(0))|\leqq\sum_{j=0}^{n}|L_n^{(k)}(p_j;q^{(n)}(0))|\,|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|\\&\leqq(n+1)\max_{(j)}|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|,\end{aligned}$$

and therefore

$$\Phi(p;0)\leqq\varliminf_{n\to\infty}\sqrt[n]{\max_{(j)}|L_n^{(j)}(p;p_\varepsilon^{(n)}(\lambda_1))|}. \tag{19}$$

Since $1-\frac{\lambda_2}{\lambda_1}$ is <0, it follows from (18) and (19) that

$$\Phi(p;\lambda_2)\leqq[\Phi(p;\lambda_1)]^{\frac{\lambda_2}{\lambda_1}}[\Phi(p;0)]^{1-\frac{\lambda_2}{\lambda_1}}.$$

Therefore

$$\left[\frac{\Phi(p;\lambda_2)}{\Phi(p;0)}\right]^{\frac{1}{\lambda_2}}\left[\frac{\Phi(p;\lambda_1)}{\Phi(p;0)}\right]^{\frac{1}{\lambda_1}}. \tag{20}$$

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[1] When $E<\{|Q_k(p)|=c>0\}$ then $E=\widetilde{E}$.

From (20) and from the Property 9 follows the existence of the limit (17). The function $\Psi(p)$ satisfies the inequalities

$$\exp m \leqq \Psi(p) \leqq \exp M, \qquad p \neq 0.$$

Suppose E is contained in the set $\{|Q_k(p)| = c > 0\}$ and satisfies the condition (H). The function

$$\left[\frac{\Phi(p;\lambda)}{\Phi(p;0)}\right]^{\frac{1}{\lambda}}$$

is continuous at every point $p \neq 0$. At every point $p_0 \in E_\lambda^*$ we have $\Phi(p_0; 0) = 1$ and $\Phi(p_0; \lambda) = \exp[\lambda f(p_0)]$. Therefore $\Psi(p_0) = \exp f(p_0)$. This equality holds at every point $p_0 \in \sum_{\lambda>0} E_\lambda^*$. In fact, suppose $p_0 \in E_{\lambda_1}^*$; then

$$\Psi(p_0) \geqq \left[\frac{\Phi(p_0;\lambda_1)}{\Phi(p_0;0)}\right]^{\frac{1}{\lambda_1}} = \exp f(p_0).$$

On the other hand $\left[\frac{\Phi(p_0;\lambda)}{\Phi(p_0;0)}\right]^{\frac{1}{\lambda}} \leqq \exp f(p_0)$ for all $\lambda > 0$ and therefore

$$\Psi(p_0) \leqq \exp f(p_0).$$

On every analytic plane which passes through the origin and an arbitrary point $p_0 \in \sum_{\lambda>0} E_\lambda^*$ we have

$$\Psi(p) = \exp f(p_0)\ (= \text{const.}), \qquad p \neq 0.$$

Remark. In the generalized Nevanlinna theory, as developed by BERGMAN, the main problem which arises consists in defining functions of extended class (see the Introduction) and investigating their properties.

In the present paper we showed that the classical results obtained previously by LEJA permit a new construction of the extended class based on the method of extremal points.

One can show that use of this method provides new possibilities for deriving bounds for functions of BERGMAN's extended class. In this way our results can be used to obtain bounds for meromorphic functions of two complex variables. One obtains relations between the growth of analytic functions of two variables and properties of their a-surfaces (*i.e.*, surfaces where the analytic function $f(z, w)$ assumes the value $f(z, w) = a$).

This paper was done under a contract between the U.S.National Science Foundation and Stanford University.

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(Received November 13, 1959)

Zur numerischen Behandlung von Anfangswertproblemen partieller hyperbolischer Differentialgleichungen zweiter Ordnung in zwei unabhängigen Veränderlichen

*I. Das charakteristische Anfangswertproblem**

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Vorgelegt von W. Tollmien

1. Einleitung

Für die numerische Behandlung von Anfangswertproblemen gewöhnlicher Differentialgleichungen sind die sogenannten Differenzenschemaverfahren bekannt und bewährt. Sie sind unter den gegenwärtig bekannten Verfahren wohl im allgemeinen die genauesten. Der einfache Grundgedanke aller Differenzenschemaverfahren ist die Umwandlung der Differentialgleichung mit den Anfangsbedingungen in Integralformen. Die Integranden dieser Integralformen werden dann durch Polynome approximiert [*5*].

Der Gedanke liegt nahe, diese Methode auf Anfangswertprobleme partieller Differentialgleichungen zu übertragen, wobei man in bestimmten Fällen und unter entsprechenden Voraussetzungen erwarten darf, daß hierdurch ähnlich gute Ergebnisse erzielt werden können. Im folgenden wird dieser Gedanke auf Anfangswertprobleme der Differentialgleichung

$$\frac{\partial^2 z}{\partial x\,\partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

übertragen. Eine Darstellung in der vorliegenden Form scheint bisher noch nicht vorhanden zu sein. In diesem I. Teil der Arbeit soll das charakteristische Anfangswertproblem behandelt werden, während die numerische Behandlung des Cauchy-Problems dem II. Teil vorbehalten bleiben soll. Alle Betrachtungen werden ausschließlich im Reellen durchgeführt.

Vor einiger Zeit hat J. Diaz [*9*] das charakteristische Anfangswertproblem der Differentialgleichung $z_{xy} = f(x, y, z, z_x, z_y)$ mit Hilfe einer dem Euler-Cauchyschen Polygonzugverfahren analogen Methode behandelt, wobei er auf deren Wert für die numerische Anwendung hinweist. Während wir die Anfangswerte auf den Charakteristiken (und im II. Teil der Arbeit beim Cauchy-Problem auf einer Kurve) im allgemeinen in analytischer Form vorgeben müssen, ist das Verfahren von Diaz immer anwendbar, wenn nur diskrete Anfangswerte vorgegeben sind. Vom numerischen Standpunkt aus gesehen scheinen jedoch zwischen

* 1. Teil einer gekürzten Fassung der 1958 von der Fakultät für Natur- und Geisteswissenschaften der Bergakademie Clausthal angenommenen Dissertation des Verfassers; Referenten: Prof. Dr. H. König und Prof. Dr. H. Menzel.

dem Verfahren von DIAZ und den hier mitgeteilten Verfahren vor allem hinsichtlich der Genauigkeit ähnliche Unterschiede zu bestehen wie zwischen dem Euler-Cauchy-Verfahren und den Differenzenschemaverfahren für gewöhnliche Differentialgleichungen.

In Anlehnung an die Differenzenschemaverfahren werden Extrapolations- und Interpolationsverfahren beschrieben, von denen sich die Interpolationsverfahren durch höhere Genauigkeit auszeichnen. Man erhält stets diskrete Lösungswerte. In den Nummern 2. bis 5. werden die Vorschriften für die praktische Rechnung — Extrapolationsverfahren, Interpolationsverfahren für das dort näher beschriebene „Anfangsfeld" und für die „fortlaufende Rechnung" — entwickelt. Bei den Interpolationsverfahren werden die Näherungen iterativ ermittelt. Die Nummern 6. bis 8. enthalten Kriterien für die Konvergenz dieser Iterationen und das Prinzip einer Fehlerabschätzung für die Werte des Anfangsfeldes. In 9. findet sich ein Beispiel.

Die wichtige Frage nach der numerischen Stabilität der mitgeteilten Verfahren wird hier nicht beantwortet. Die Betrachtungen darüber sollen einer späteren Arbeit vorbehalten bleiben.

2. Ausgangsgleichungen

Es sei das charakteristische Anfangswertproblem

$$\begin{gathered}\frac{\partial^2 z}{\partial x\,\partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right),\\ z(x,\eta) = \sigma(x), \quad z(\xi, y) = \tau(y), \quad \text{mit} \quad \sigma(\xi) = \tau(\eta)\end{gathered} \tag{1}$$

zur Lösung vorgelegt. Bekanntlich lassen sich alle hyperbolischen Differentialgleichungen 2. Ordnung in zwei unabhängigen Veränderlichen mit linearem Hauptteil auf die Normalform (1) transformieren. Die praktische Durchführung dieser Transformation wird manchmal nicht leicht sein. Darin liegt die einzige Schwierigkeit bei der Benutzung der im folgenden zu entwickelnden Näherungsverfahren.

Die Frage nach der Existenz und Eindeutigkeit der Lösung des Problems (1) beantwortet der [2] [3]

Satz 1. *In dem Rechteck*

$$\mathfrak{R}: \quad a_1 < x < b_1, \qquad c_1 < y < d_1, \qquad z, u, v \quad \textit{beliebig},$$

sei die Funktion $f(x, y, z, u, v)$ stetig, in jedem abgeschlossenen Teilgebiet g von $\mathfrak{R}$ beschränkt und erfülle dort eine Lipschitz-Bedingung

$$|f(x, y, z_2, u_2, v_2) - f(x, y, z_1, u_1, v_1)| \leqq M\left(|z_2 - z_1| + |u_2 - u_1| + |v_2 - v_1|\right).$$

Es seien in den Intervallen $a_1 < x < b_1$, $c_1 < y < d_1$ ferner die stetig differenzierbaren Funktionen $\sigma(x)$ und $\tau(y)$ vorgegeben und (ξ, η) sei ein innerer Punkt von $\mathfrak{R}$ mit $\sigma(\xi) = \tau(\eta)$. Dann hat das Problem (1) in $\mathfrak{R}$ die eindeutig bestimmte Lösung $z = \psi(x, y)$ mit $\psi(x, \eta) = \sigma(x)$ und $\psi(\xi, y) = \tau(y)$.

Es ist zu beachten, daß durch die Vorgabe der Anfangswerte $\sigma(x)$ und $\tau(y)$ auf den Charakteristiken $y - \eta$ und $x = \xi$ die Lösung des Problems schon eindeutig bestimmt ist. Man erhält sofort

$$\frac{\partial \psi(x,\eta)}{\partial x} = \sigma'(x), \qquad \frac{\partial \psi(\xi, y)}{\partial y} = \tau'(y).$$

Die Differentialgleichung (1) stellt längs $x=\xi$ eine gewöhnliche Differentialgleichung für $\frac{\partial\psi(\xi,y)}{\partial x}$ dar, da $\frac{\partial\psi(\xi,y)}{\partial y}$ dort bekannt ist. Ebenso erhält man eine gewöhnliche Differentialgleichung für $\frac{\partial\psi(x,\eta)}{\partial y}$ längs $y=\eta$. Die Lösungen dieser Gleichungen existieren und sind eindeutig. Damit sind die z, z_x, z_y auf den Anfangsgeraden festgelegt.

Das Problem (1) genüge den Voraussetzungen des Satz 1. Mit den üblichen Abkürzungen $z_{xy}=s$, $z_x=p$, $z_y=q$ ergibt sich als Lösung:

$$\begin{aligned}
z(x,y) &= z(x,\eta)+z(\xi,y)-z(\xi,\eta)+\int_\xi^x\int_\eta^y f(u,v,z,p,q)\,dv\,du,\\
p(x,y) &= p(x,\eta)+\int_\eta^y f(x,v,z,p,q)\,dv,\\
q(x,y) &= q(\xi,v)+\int_\xi^x f(u,y,z,p,q)\,du.
\end{aligned}\tag{2}$$

Die Integranden in den letzten beiden Gleichungen sind Veränderliche nur von x allein bzw. von y allein.

Die Intervalle $a_1<a\leqq x\leqq b<b_1$, $c_1<c\leqq y\leqq d<d_1$ werden unterteilt durch

$$\begin{aligned}
a=x_0<x_1<x_2<\cdots<x_j=b,\\
c=y_0<y_1<y_2<\cdots<y_l=d.
\end{aligned}\tag{3}$$

Zieht man durch die Teilpunkte der Intervalle Parallelen zu den Koordinatenachsen, so wird das Rechteck $a\leqq x\leqq b$, $c\leqq y\leqq d$ in $j\,l$ Teilrechtecke unterteilt.

3. Extrapolationsverfahren

Die Rechnung sei schon bis zu Abszissen $x=x_r$, $y=y_s$ fortgeschritten und daher Näherungen

$$\tilde z(x_\mu,y_\nu)=z_{\mu,\nu},\quad \tilde p(x_\mu,y_\nu)=p_{\mu,\nu},\quad \tilde q(x_\mu,y_\nu)=q_{\mu,\nu},$$
$$\tilde f(x_\mu,y_\nu,z_{\mu,\nu},p_{\mu,\nu},q_{\mu,\nu})=f_{\mu,\nu},$$
$$(\mu=0,1,\ldots,r;\ \nu=0,1,\ldots,s;\ r<j,\ s<l)$$

bekannt. Es wird nun in (2) $f(x,y,z,p,q)$ durch das Polynom $P_{m,n}(x,y)$ ersetzt, das an den Stellen $(x_{r-\mu},y_{s-\nu})$ die Werte $f_{r-\mu,s-\nu}$ annimmt, $(\mu=0,1,\ldots,m;$ $\nu=0,1,\ldots,n;\ m\leqq r,\ n\leqq s)$.

Sei stets bei allen folgenden Betrachtungen

$$x_{\mu+1}-x_\mu=h,\qquad y_{\nu+1}-y_\nu=k,\qquad (\mu=0,1,\ldots,j-1;\ \nu=0,1,\ldots,l-1),$$

dann ist mit der Transformation [1]

$$\frac{x-x_r}{h}=u,\qquad \frac{y-y_s}{k}=v$$

und der Abkürzung $W^{(-\varrho)}=w(w+1)(w+2)\ldots(w+\varrho-1)$:

$$P_{m,n}(x,y)=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\frac{U^{(-\mu)}}{\mu!}\,\frac{V^{(-\nu)}}{\nu!}\,\nabla_x^\mu\,\nabla_y^\nu f_{r,s}.\tag{4a}$$

Es wird ferner $f(x_{r+1}, y, z, p, q)$ durch das Polynom $\overline{P}_n(y)$ ersetzt, das an den Stellen $(x_{r+1}, y_{s-\nu})$ die Werte $f_{r+1, s-\nu}$ annimmt, schließlich $f(x, y_{s+1}, z, p, q)$ durch das Polynom $\overline{\overline{P}}_m(x)$, das an den Stellen $(x_{r-\mu}, y_{s+1})$ die Werte $f_{r-\mu, s+1}$ annimmt:

$$\overline{P}_n(y) = \sum_{\nu=0}^{n} \frac{V^{(-\nu)}}{\nu!} \nabla_y^\nu f_{r+1, s}, \tag{4b}$$

$$\overline{\overline{P}}_m(x) = \sum_{\mu=0}^{m} \frac{U^{(-\mu)}}{\mu!} \nabla_x^\mu f_{r, s+1}. \tag{4c}$$

Nach (2) erhält man dann mit $\xi = x_r$, $\eta = y_s$ die Näherungswerte:

$$\begin{aligned}
z_{r+1, s+1} &= z_{r+1, s} + z_{r, s+1} - z_{r, s} + h k \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \beta_\mu \beta_\nu \nabla_x^\mu \nabla_y^\nu f_{r, s}, \\
p_{r+1, s+1} &= p_{r+1, s} + k \sum_{\nu=0}^{n} \beta_\nu \nabla_y^\nu f_{r+1, s}, \\
q_{r+1, s+1} &= q_{r, s+1} + h \sum_{\mu=0}^{m} \beta_\mu \nabla_x^\mu f_{r, s+1}, \\
f_{r+1, s+1} &= f(x_{r+1}, y_{s+1}, z_{r+1, s+1}, p_{r+1, s+1}, q_{r+1, s+1}),
\end{aligned} \tag{5}$$

mit

$$\beta_\varrho = \frac{1}{\varrho!} \int_0^1 U^{(-\varrho)} \, du = \frac{1}{\varrho!} \int_0^1 u(u+1)(u+2)\dots(u+\varrho-1)\, du. \tag{6}$$

Tabelle 1 [5]. *Gewichte β_ϱ*

ϱ	0	1	2	3	4	5
β_ϱ	$\frac{1440}{1440}$	$\frac{720}{1440}$	$\frac{600}{1440}$	$\frac{540}{1440}$	$\frac{502}{1440}$	$\frac{475}{1440}$

Setzt man $f(x, y, z, p, q) = F(x, y)$, so gilt für die exakte Lösung der Differentialgleichung:

$$\begin{aligned}
z(x_{r+1}, y_{s+1}) &= z(x_{r+1}, y_s) + z(x_r, y_{s+1}) - z(x_r, y_s) + \\
&\quad + h k \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \beta_\mu \beta_\nu \nabla_x^\mu \nabla_y^\nu F(x_r, y_s) + S_{m+1, n+1}, \\
p(x_{r+1}, y_{s+1}) &= p(x_{r+1}, y_s) + k \sum_{\nu=0}^{n} \beta_\nu \nabla_y^\nu F(x_{r+1}, y_s) + \overline{S}_{n+1}, \\
q(x_{r+1}, y_{s+1}) &= q(x_r, y_{s+1}) + h \sum_{\mu=0}^{m} \beta_\mu \nabla_x^\mu F(x_r, y_{s+1}) + \overline{\overline{S}}_{m+1}.
\end{aligned} \tag{7}$$

Die Kubatur- und Quadraturfehler können abgeschätzt werden [1]:

$$\begin{aligned}
|S_{m+1, n+1}| &\leqq k h^{m+2} |\beta_{m+1}| \left| \frac{\partial^{m+1}}{\partial x^{m+1}} F \right|_{\text{Max}} + h k^{n+2} |\beta_{n+1}| \left| \frac{\partial^{n+1}}{\partial y^{n+1}} F \right|_{\text{Max}} + \\
&\quad + h^{m+2} k^{n+2} |\beta_{m+1}| \, |\beta_{n+1}| \left| \frac{\partial^{m+n+2}}{\partial x^{m+1} \partial y^{n+1}} F \right|_{\text{Max}}, \\
|\overline{S}_{n+1}| &\leqq k^{n+2} |\beta_{n+1}| \left| \frac{\partial^{n+1}}{\partial y^{n+1}} F \right|_{\text{Max}}, \\
|\overline{\overline{S}}_{m+1}| &\leqq h^{m+2} |\beta_{m+1}| \left| \frac{\partial^{m+1}}{\partial x^{m+1}} F \right|_{\text{Max}}.
\end{aligned} \tag{8}$$

Mit den Formeln (5) lassen sich fortlaufend neue Näherungswerte der Lösung berechnen. Da die praktische Differenzenbildung nach 2 Veränderlichen recht umständlich ist, erscheint es zweckmäßig, statt der Differenzen direkt die Funktionswerte zu schreiben. Aus (5) erhält man dann nach kurzer Rechnung:

$$\begin{aligned}
z_{r+1,s+1} &= z_{r+1,s} + z_{r,s+1} - z_{r,s} + h\,k \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{m,\mu}\, \alpha_{n,\nu}\, f_{r-\mu,s-\nu}\,, \\
p_{r+1,s+1} &= p_{r+1,s} + k \sum_{\nu=0}^{n} \alpha_{n,\nu}\, f_{r+1,s-\nu}\,, \\
q_{r+1,s+1} &= q_{r,s+1} + h \sum_{\mu=0}^{m} \alpha_{m,\mu}\, f_{r-\mu,s+1}\,, \\
f_{r+1,s+1} &= f(x_{r+1}, y_{s+1}, z_{r+1,s+1}, p_{r+1,s+1}, q_{r+1,s+1})
\end{aligned} \tag{9}$$

mit den Gewichten

$$\alpha_{j,l} = (-1)^l \sum_{\lambda=l}^{j} \beta_\lambda \binom{\lambda}{l}. \tag{10}$$

Für die exakte Lösung der Differentialgleichung gelten entsprechende Gleichungen mit den Abschätzungen (8) für die Kubatur- und Quadraturfehler.

Tabelle 2 [5]. *Gewichte* $\alpha_{j,l}$

l	j = 0	1	2	3	4	5
0	1	$\frac{3}{2}$	$\frac{23}{12}$	$\frac{55}{24}$	$\frac{1901}{720}$	$\frac{4277}{1440}$
1		$-\frac{1}{2}$	$-\frac{16}{12}$	$-\frac{59}{24}$	$-\frac{2774}{720}$	$-\frac{7923}{1440}$
2			$\frac{5}{12}$	$\frac{37}{24}$	$\frac{2616}{720}$	$\frac{9982}{1440}$
3				$-\frac{9}{24}$	$-\frac{1274}{720}$	$-\frac{7298}{1440}$
4					$\frac{251}{720}$	$\frac{2877}{1440}$
5						$-\frac{475}{1440}$

Die Approximation der Integranden in (2) kann offenbar auf sehr verschiedene Arten vorgenommen werden. Es soll darauf aber nicht näher eingegangen werden, zumal durch Extrapolation im allgemeinen nur „Rohwerte" gefunden werden sollen, die dann durch Interpolation verbessert werden.

4. Berechnung des Anfangsfeldes

Um die Formeln aus 3. und die im nächsten Abschnitt 5. entwickelten Interpolationsformeln anwenden zu können, benötigt man schon entlang der Anfangsgeraden ein „Anfangsfeld" von Werten. Zu seiner Berechnung können alle bekannten, hinreichend genauen Verfahren dienen, wie z.B. die verbesserten Differenzenverfahren und Reihenentwicklungen. Im folgenden wird ein Verfahren vorgeschlagen, das unter den Voraussetzungen des Satz 1 in 2. stets anwendbar ist.

Auf den Charakteristiken $x = x_0$ und $y = y_0$ seien die Anfangswerte $z(x_0, y)$ und $z(x, y_0)$ vorgegeben. Dann kennt man auch die p und q längs der Anfangsgeraden. Dabei sind $p(x_0, y)$ und $q(x, y_0)$ eindeutige Lösungen gewöhnlicher Differentialgleichungen. Gelingt es nicht, diese Gleichungen exakt zu lösen, dann empfehlen sich hier zur numerischen Lösung besonders die Differenzenschemaverfahren. (Siehe z.B. [5]). Man verschafft sich nun zuerst Rohwerte $z^{[0]}_{r,s}$, $p^{[0]}_{r,s}$, $q^{[0]}_{r,s}$, $f^{[0]}_{r,s}$, indem man z.B. die Formeln (9) benutzt und $r-1$ statt m, $s-1$ statt n setzt, $(r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n)$:

$$\left.\begin{aligned} z^{[0]}_{r,s} &= z^{[0]}_{r,s-1} + z^{[0]}_{r-1,s} - z^{[0]}_{r-1,s-1} + hk \sum_{\mu=0}^{r-1} \sum_{\nu=0}^{s-1} \alpha_{r-1,\mu}\, \alpha_{s-1,\nu}\, f^{[0]}_{r-1-\mu,\, s-1-\nu}, \\ p^{[0]}_{r,s} &= p^{[0]}_{r,s-1} + k \sum_{\nu=0}^{s-1} \alpha_{s-1,\nu}\, f^{[0]}_{r,s-1-\nu}, \\ q^{[0]}_{r,s} &= q^{[0]}_{r-1,s} + h \sum_{\mu=0}^{r-1} \alpha_{r-1,\mu}\, f^{[0]}_{r-1-\mu,s}, \\ f^{[0]}_{r,s} &= f(x_r, y_s, z^{[0]}_{r,s}, p^{[0]}_{r,s}, q^{[0]}_{r,s}). \end{aligned}\right\} \tag{11}$$

Dabei ist

$$\left.\begin{aligned} z^{[0]}_{j,l} &= z(x_j, y_l) \\ p^{[0]}_{j,l} &= p(x_j, y_l) \\ q^{[0]}_{j,l} &= q(x_j, y_l) \\ f^{[0]}_{j,l} &= F(x_j, y_l) \end{aligned}\right\} \quad \text{für} \quad j = 0 \quad \text{oder} \quad l = 0 \quad \text{oder} \quad j = l = 0. \tag{12}$$

Die so berechneten Rohwerte werden nun durch ein Interpolationsverfahren verbessert: Mit [1]

$$P^{(\lambda)}(z) = \prod_{\nu=0}^{\lambda} (z - z_\nu), \qquad P^{(\lambda)}_{\alpha}(z) = \frac{P^{(\lambda)}(z)}{z - z_\alpha}$$

ist

$$P_{m,n}(x, y) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \frac{P^{(m)}_{\mu}(x)}{P^{(m)}_{\mu}(x_\mu)} \frac{P^{(n)}_{\nu}(y)}{P^{(n)}_{\nu}(y_\nu)} f_{\mu,\nu} \tag{13}$$

das Polynom, das an den Stellen (x_μ, y_ν) die Werte $f_{\mu,\nu}$ annimmt, $(\mu = 0, 1, \ldots, m;$ $\nu = 0, 1, \ldots, n)$. Mit der Transformation

$$x = x_0 + uh, \qquad y = y_0 + vk$$

wird daraus

$$P_{m,n}(x, y) = Q_{m,n}(u, v) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \frac{Q^{(m)}_{\mu}(u)}{Q^{(m)}_{\mu}(\mu)} \frac{Q^{(n)}_{\nu}(v)}{Q^{(n)}_{\nu}(\nu)} \cdot f_{\mu,\nu}, \tag{14}$$

mit

$$Q^{(\lambda)}_{\alpha}(w) = \frac{1}{w - \alpha} \prod_{\nu=0}^{\lambda} (w - \nu).$$

Ferner ist

$$\overline{Q}_n(v) = \sum_{\nu=0}^{n} \frac{Q^{(n)}_{\nu}(v)}{Q^{(n)}_{\nu}(\nu)} f_{r,\nu} \tag{15a}$$

das Polynom, das an den Stellen (x_r, y_ν) die Werte $f_{r,\nu}$ und

$$\overline{\overline{Q}}_m(u) = \sum_{\mu=0}^{m} \frac{Q^{(m)}_{\mu}(u)}{Q^{(m)}_{\mu}(\mu)} f_{\mu,s} \tag{15b}$$

das Polynom, das an den Stellen (x_μ, y_s) die Werte $f_{\mu,s}$ annimmt. Setzt man in (2) $\xi = x_0$, $\eta = y_0$, ersetzt die Integranden durch die Polynome (14) und (15), so erhält man die Interpolationsformeln:

$$\left.\begin{aligned} z_{r,s} &= z_{r,0} + z_{0,s} - z_{0,0} + hk \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \gamma^m_{r,\mu} \gamma^n_{s,\nu} f_{\mu,\nu}, \\ p_{r,s} &= p_{r,0} + k \sum_{\nu=0}^{n} \gamma^n_{s,\nu} f_{r,\nu}, \\ q_{r,s} &= q_{0,s} + h \sum_{\mu=0}^{m} \gamma^m_{r,\mu} f_{\mu,s}, \\ f_{r,s} &= f(x_r, y_s, z_{r,s}, p_{r,s}, q_{r,s}), \qquad (r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n), \end{aligned}\right\} \tag{16}$$

mit

$$\gamma^j_{\varkappa,\lambda} = \int_0^\varkappa \frac{Q^j_\lambda(w)}{Q^j_\lambda(\lambda)} dw = \frac{(-1)^{j-\lambda}}{\lambda!(j-\lambda)!} \int_0^\varkappa w(w-1)\ldots(w-\lambda+1)(w-\lambda-1)\ldots(w-j)\,dw. \tag{17}$$

Tabelle 3 [6]. *Gewichte* $\gamma^j_{\varkappa,\lambda}$

Die Zahlenwerte für die Matrizen

$$\begin{pmatrix} \gamma^j_{1,0} & \gamma^j_{1,1} & \cdots & \gamma^j_{1,j} \\ \gamma^j_{2,0} & \gamma^j_{2,1} & \cdots & \gamma^j_{2,j} \\ \vdots & \vdots & & \vdots \\ \gamma^j_{j,0} & \gamma^j_{j,1} & \cdots & \gamma^j_{j,j} \end{pmatrix}$$

sind:

$j = 1$:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$j = 2$:

$$\begin{pmatrix} \frac{5}{12} & \frac{8}{12} & -\frac{1}{12} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} \end{pmatrix},$$

$j = 3$:

$$\begin{pmatrix} \frac{9}{24} & \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{3}{8} & \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{pmatrix}$$

Für die exakte Lösung der Differentialgleichung gilt:

$$\left.\begin{aligned} z(x_r, y_s) &= z(x_r, y_0) + z(x_0, y_s) - z(x_0, y_0) + hk \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \gamma^m_{r,\mu} \gamma^n_{s,\nu} F(x_\mu, y_\nu) + S^{r,s}_{m+1,n+1}, \\ p(x_r, y_s) &= p(x_r, y_0) + k \sum_{\nu=0}^{n} \gamma^n_{s,\nu} F(x_r, y_\nu) + \bar{S}^{r,s}_{n+1}, \\ q(x_r, y_s) &= q(x_0, y_s) + h \sum_{\mu=0}^{m} \gamma^m_{r,\mu} F(x_\mu, y_s) + \bar{\bar{S}}^{r,s}_{m+1}. \end{aligned}\right\} \tag{18}$$

Die Abschätzung der Kubatur- und Quadraturfehler ist hier unbequem; man erhält nach einiger Rechnung [1]:

$$\begin{aligned}
\left|S^{r,s}_{m+1,n+1}\right| &\leqq s\,k\,h^{m+2}\left|\frac{\partial^{m+1}}{\partial x^{m+1}}F\right|_{\mathrm{Max}}\sum_{j=0}^{r-1}\left|\gamma^{m+1}_{j+1,m+1}-\gamma^{m+1}_{j,m+1}\right|+\\
&+r\,h\,k^{n+2}\left|\frac{\partial^{n+1}}{\partial y^{n+1}}F\right|_{\mathrm{Max}}\sum_{l=0}^{s-1}\left|\gamma^{n+1}_{l+1,n+1}-\gamma^{n+1}_{l,n+1}\right|+\\
&+h^{m+2}k^{n+2}\left|\frac{\partial^{m+n+2}}{\partial x^{m+1}\partial y^{n+1}}F\right|_{\mathrm{Max}}\sum_{j=0}^{r-1}\sum_{l=0}^{s-1}\left|\gamma^{m+1}_{j+1,m+1}-\gamma^{m+1}_{j,m+1}\right|\left|\gamma^{n+1}_{l+1,n+1}-\gamma^{n+1}_{l,n+1}\right|,
\end{aligned}\tag{19}$$

$$\left|\bar S^{r,s}_{n+1}\right|\leqq k^{n+2}\left|\frac{\partial^{n+1}}{\partial y^{n+1}}F\right|_{\mathrm{Max}}\sum_{l=0}^{s-1}\left|\gamma^{n+1}_{l+1,n+1}-\gamma^{n+1}_{l,n+1}\right|,$$

$$\left|\bar{\bar S}^{r,s}_{m+1}\right|\leqq h^{m+2}\left|\frac{\partial^{m+1}}{\partial x^{m+1}}F\right|_{\mathrm{Max}}\sum_{j=0}^{r-1}\left|\gamma^{m+1}_{j+1,m+1}-\gamma^{m+1}_{j,m+1}\right|.$$

Es sei vorausgesetzt, daß die in (19) auftretenden Ableitungen beschränkt sind. Dann ist mit $k=h$, $m=n$ und für $h\to 0$:

$$S^{r,s}_{m+1,n+1}(h)=o(h^{m+2}),\qquad \bar S^{r,s}_{n+1}(h)=o(h^{m+1}),\qquad \bar{\bar S}^{r,s}_{m+1}(h)=o(h^{m+1}).$$

Nun zur Anwendung der Formeln (16): Man kennt bereits Rohwerte $z^{[0]}_{\mu,\nu}$, $p^{[0]}_{\mu,\nu}$, $q^{[0]}_{\mu,\nu}$, $f^{[0]}_{\mu,\nu}$. Erste Näherungen werden dann berechnet mit Hilfe der Iterationsvorschriften:

$$\begin{aligned}
z^{[1]}_{r,s}&=z_{r,0}+z_{0,s}-z_{0,0}+h\,k\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\gamma^{m}_{r,\mu}\gamma^{n}_{s,\nu}f^{[0]}_{\mu,\nu},\\
p^{[1]}_{r,s}&=p_{r,0}+k\sum_{\nu=0}^{n}\gamma^{n}_{s,\nu}f^{[0]}_{r,\nu},\\
q^{[1]}_{r,s}&=q_{0,s}+h\sum_{\mu=0}^{m}\gamma^{m}_{r,\mu}f^{[0]}_{\mu,s},\\
f^{[1]}_{r,s}&=f(x_r,y_s,z^{[1]}_{r,s},p^{[1]}_{r,s},q^{[1]}_{r,s}),\qquad (r=1,2,\ldots,m;\ s=1,2,\ldots,n),
\end{aligned}\tag{20}$$

mit der Schreibweise (12).

Bei der Berechnung weiterer Näherungen vereinfachen sich die Vorschriften (20): Die $(\lambda+1)$-ten Näherungen sind:

$$\begin{aligned}
z^{[\lambda+1]}_{r,s}&=z^{[\lambda]}_{r,s}+h\,k\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}\gamma^{m}_{r,\mu}\gamma^{n}_{s,\nu}(f^{[\lambda]}_{\mu,\nu}-f^{[\lambda-1]}_{\mu,\nu}),\\
p^{[\lambda+1]}_{r,s}&=p^{[\lambda]}_{r,s}+k\sum_{\nu=1}^{n}\gamma^{n}_{s,\nu}(f^{[\lambda]}_{r,\nu}-f^{[\lambda-1]}_{r,\nu}),\\
q^{[\lambda+1]}_{r,s}&=q^{[\lambda]}_{r,s}+h\sum_{\mu=1}^{m}\gamma^{m}_{r,\mu}(f^{[\lambda]}_{\mu,s}-f^{[\lambda-1]}_{\mu,s}),\\
f^{[\lambda+1]}_{r,s}&=f(x_r,y_s,z^{[\lambda+1]}_{r,s},p^{[\lambda+1]}_{r,s},q^{[\lambda+1]}_{r,s}),
\end{aligned}\tag{21}$$

für $\lambda\geqq 1$.

Hinreichende Bedingungen für die Konvergenz dieses Verfahrens werden in 6. gegeben.

Mit Hilfe der Formeln (11), (20) und (21) berechnet man zunächst Näherungswerte über dem in Abb. 1 (für $m=n=2$) mit „1“ bezeichneten Rechteck. Um die Verfahren der fortlaufenden Rechnung anwenden zu können, benötigt man noch Werte über den in Abb. 1 längs $y=y_0$ mit „2“, „3“, ... und längs $x=x_0$ mit „2'“, „3'“, ... bezeichneten Rechtecken. Die Formeln (11), (20) und (21) dienen zur Berechnung auch dieser Werte, man hat lediglich

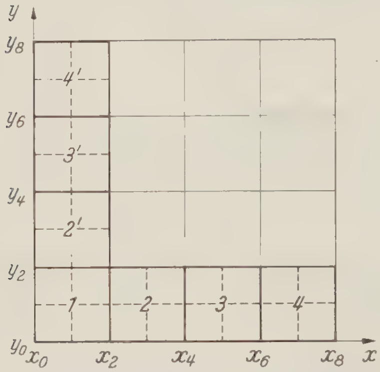

Abb. 1. Zur Berechnung des Anfangsfeldes

bei der Rechnung längs $y=y_0$ für die $z_{\varrho,\lambda}$, $p_{\varrho,\lambda}$, $q_{\varrho,\lambda}$, $f_{\varrho,\lambda}$ den Index ϱ durch

$$Mm+\varrho, \quad (M=1,2,\ldots;\ \varrho=1,2,\ldots,m),$$

bei der Rechnung längs $x=x_0$ für die $z_{\varrho,\lambda}$, $p_{\varrho,\lambda}$, $q_{\varrho,\lambda}$, $f_{\varrho,\lambda}$ den Index λ durch

$$Nn+\lambda, \quad (N=1,2,\ldots;\ \lambda=1,2,\ldots,n),$$

zu ersetzen.

5. Interpolationsverfahren für die fortlaufende Rechnung

Man ersetzt hier $f(x,y,z,p,q)$ in (2) durch das Polynom $P^*_{m,n}(x,y)$, das an den Stellen $(x_{r+1-\mu}, y_{s+1-\nu})$, $(\mu=0,1,\ldots,m;\ \nu=0,1,\ldots,n)$, die Werte $f_{r+1-\mu,s+1-\nu}$ annimmt. Entsprechend werden $f(x,v,z,p,q)$ und $f(u,y,z,p,q)$ ersetzt. Da die Herleitung der Interpolationsformeln ganz analog der Herleitung von (9) erfolgt, können wir uns sehr kurz fassen. Mit der Transformation

$$\frac{x-x_r}{h}=u, \qquad \frac{y-y_s}{k}=v$$

und $\xi=x_r$, $\eta=y_s$ erhält man zunächst:

$$\left.\begin{aligned}
z_{r+1,s+1} &= z_{r+1,s}+z_{r,s+1}-z_{r,s}+h\,k\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\beta^*_\mu\,\beta^*_\nu\,\nabla^\mu_x\,\nabla^\nu_y\,f_{r+1,s+1},\\
p_{r+1,s+1} &= p_{r+1,s}+k\sum_{\nu=0}^{n}\beta^*_\nu\,\nabla^\nu_y\,f_{r+1,s+1},\\
q_{r+1,s+1} &= q_{r,s+1}+h\sum_{\mu=0}^{m}\beta^*_\mu\,\nabla^\mu_x\,f_{r+1,s+1},\\
f_{r+1,s+1} &= f(x_{r+1},y_{s+1},z_{r+1,s+1},p_{r+1,s+1},q_{r+1,s+1}),
\end{aligned}\right\} \tag{22}$$

mit

$$\beta^*_l=\frac{1}{l!}\int_0^1(u-1)\,u(u+1)\ldots(u+l-2)\,du. \tag{23}$$

Tabelle 4 [5]. *Gewichte* β^*_l

l	0	1	2	3	4	5
β^*_l	$\frac{720}{720}$	$-\frac{360}{720}$	$-\frac{60}{720}$	$-\frac{30}{720}$	$-\frac{19}{720}$	$-\frac{27}{720}$

Die in (22) auftretenden Differenzen werden wieder durch Funktionswerte ersetzt. Auf der rechten Seite der dann entstehenden Vorschriften kennt man den Wert $f_{r+1,s+1}$ nicht, sondern nur einen Rohwert $f^{[0]}_{r+1,s+1}$. Man wird daher

die Näherungen wieder iterativ ermitteln mit Hilfe der Gleichungen:

$$
\begin{aligned}
z^{[1]}_{r+1,s+1} &= z_{r+1,s} + z_{r,s+1} - z_{r,s} + \\
&\quad + h k \left[\alpha^*_{m,0}\, \alpha^*_{n,0}\, f^{[0]}_{r+1,s+1} + \sum_{\substack{\mu=0 \\ \mu,\nu \neq 0,0}}^{m} \sum_{\nu=0}^{n} \alpha^*_{m,\mu}\, \alpha^*_{n,\nu}\, f_{r+1-\mu,s+1-\nu} \right], \\
p^{[1]}_{r+1,s+1} &= p_{r+1,s} + k \left[\alpha^*_{n,0}\, f^{[0]}_{r+1,s+1} + \sum_{\nu=1}^{n} \alpha^*_{n,\nu}\, f_{r+1,s+1-\nu} \right], \\
q^{[1]}_{r+1,s+1} &= q_{r,s+1} + h \left[\alpha^*_{m,0}\, f^{[0]}_{r+1,s+1} + \sum_{\mu=1}^{m} \alpha^*_{m,\mu}\, f_{r+1-\mu,s+1} \right], \\
f^{[1]}_{r+1,s+1} &= f\left(x_{r+1}, y_{s+1}, z^{[1]}_{r+1,s+1}, p^{[1]}_{r+1,s+1}, q^{[1]}_{r+1,s+1}\right),
\end{aligned}
\tag{24}
$$

mit

$$
\alpha^*_{j,l} = (-1)^l \sum_{\lambda=l}^{j} \binom{\lambda}{l} \beta^*_\lambda, \qquad (l = 0, 1, 2, \ldots). \tag{25}
$$

Für $\lambda \geqq 1$ vereinfachen sich diese Formeln beträchtlich:

$$
\begin{aligned}
z^{[\lambda+1]}_{r+1,s+1} &= z^{[\lambda]}_{r+1,s+1} + h k\, \alpha^*_{m,0}\, \alpha^*_{n,0} \left(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}\right), \\
p^{[\lambda+1]}_{r+1,s+1} &= p^{[\lambda]}_{r+1,s+1} + k\, \alpha^*_{n,0} \left(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}\right), \\
q^{[\lambda+1]}_{r+1,s+1} &= q^{[\lambda]}_{r+1,s+1} + h\, \alpha^*_{m,0} \left(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}\right), \\
f^{[\lambda+1]}_{r+1,s+1} &= f\left(x_{r+1}, y_{s+1}, z^{[\lambda+1]}_{r+1,s+1}, p^{[\lambda+1]}_{r+1,s+1}, q^{[\lambda+1]}_{r+1,s+1}\right).
\end{aligned}
\tag{26}
$$

Tabelle 5 [5]. *Gewichte* $\alpha^*_{j,l}$

l	j = 0	1	2	3	4	5
0	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{9}{24}$	$\frac{251}{720}$	$\frac{475}{1440}$
1		$\frac{1}{2}$	$\frac{8}{12}$	$\frac{19}{24}$	$\frac{646}{720}$	$\frac{1427}{1440}$
2			$-\frac{1}{12}$	$-\frac{5}{24}$	$-\frac{264}{720}$	$-\frac{798}{1440}$
3				$\frac{1}{24}$	$\frac{106}{720}$	$\frac{482}{1440}$
4					$-\frac{19}{720}$	$-\frac{173}{1440}$
5						$\frac{27}{1440}$

Für die exakte Lösung der Differentialgleichung gilt:

$$
\begin{aligned}
z(x_{r+1}, y_{s+1}) &= z(x_{r+1}, y_s) + z(x_r, y_{s+1}) - z(x_r, y_s) + \\
&\quad + h k \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha^*_{m,\mu}\, \alpha^*_{n,\nu}\, F(x_{r+1-\mu}, y_{s+1-\nu}) + S^*_{m+1,n+1}, \\
p(x_{r+1}, y_{s+1}) &= p(x_{r+1}, y_s) + k \sum_{\nu=0}^{n} \alpha^*_{n,\nu}\, F(x_{r+1}, y_{s+1-\nu}) + \bar{S}^*_{n+1}, \\
q(x_{r+1}, y_{s+1}) &= q(x_r, y_{s+1}) + h \sum_{\mu=0}^{m} \alpha^*_{m,\mu}\, F(x_{r+1-\mu}, y_{s+1}) + \bar{\bar{S}}^*_{m+1}.
\end{aligned}
\tag{27}
$$

Abschätzungen für die Kubatur- und Quadraturfehler sind [1]:

$$
\begin{aligned}
|S^*_{m+1,n+1}| &\leqq k\,h^{m+2}|\beta^*_{m+1}|\left|\frac{\partial^{m+1}}{\partial x^{m+1}}F\right|_{\mathrm{Max}} + h\,k^{n+2}|\beta^*_{n+1}|\left|\frac{\partial^{n+1}}{\partial y^{n+1}}F\right|_{\mathrm{Max}} + \\
&\quad + h^{m+2}k^{n+2}|\beta^*_{m+1}|\,|\beta^*_{n+1}|\left|\frac{\partial^{m+n+2}}{\partial x^{m+1}\partial y^{n+1}}F\right|_{\mathrm{Max}}, \\
|\bar{S}^*_{n+1}| &\leqq k^{n+2}|\beta^*_{n+1}|\left|\frac{\partial^{n+1}}{\partial y^{n+1}}F\right|_{\mathrm{Max}}, \\
|\bar{\bar{S}}^*_{m+1}| &\leqq h^{m+2}|\beta^*_{m+1}|\left|\frac{\partial^{m+1}}{\partial x^{m+1}}F\right|_{\mathrm{Max}}.
\end{aligned}
\tag{28}
$$

Den in (24) bei der ersten Iteration benötigten Wert $f^{[0]}_{r+1,s+1}$ kann man sich mit Hilfe der Extrapolationsformeln (9) verschaffen, oft genügt aber auch eine Schätzung dieses Wertes. Man wird auch wegen der geringen Rechenarbeit einige Iterationen mehr in Kauf nehmen, anstatt vorher durch die etwas umständliche Extrapolation einen besseren Ausgangswert $f^{[0]}_{r+1,s+1}$ zu berechnen.

Dem „Verfahren der zentralen Differenzen" für gewöhnliche Differentialgleichungen [5] entspricht folgendes Interpolationsverfahren, das wiederum als Iterationsvorschrift benutzt wird:

$$
\begin{aligned}
z^{[1]}_{r+1,s+1} &= z_{r+1,s-1} + z_{r-1,s+1} - z_{r-1,s-1} + \\
&\quad + \frac{h\,k}{9}\,[f^{[0]}_{r+1,s+1} + f_{r-1,s+1} + f_{r-1,s-1} + f_{r+1,s-1} + \\
&\quad + 4(f_{r,s+1} + f_{r-1,s} + f_{r,s-1} + f_{r+1,s}) + 16 f_{r,s}], \\
p^{[1]}_{r+1,s+1} &= p_{r+1,s-1} + \frac{k}{3}\,[f^{[0]}_{r+1,s+1} + f_{r+1,s-1} + 4 f_{r,s}], \\
q^{[1]}_{r+1,s+1} &= q_{r-1,s+1} + \frac{h}{3}\,[f^{[0]}_{r+1,s+1} + f_{r-1,s+1} + 4 f_{r,s}], \\
f^{[1]}_{r+1,s+1} &= f(x_{r+1}, y_{s+1}, z^{[0]}_{r+1,s+1}, p^{[0]}_{r+1,s+1}, q^{[0]}_{r+1,s+1}).
\end{aligned}
\tag{29}
$$

Für $\lambda \geqq 1$ vereinfachen sich die Gleichungen zu

$$
\begin{aligned}
z^{[\lambda+1]}_{r+1,s+1} &= z^{[\lambda]}_{r+1,s+1} + \frac{h\,k}{9}\,(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}), \\
p^{[\lambda+1]}_{r+1,s+1} &= p^{[\lambda]}_{r+1,s+1} + \frac{k}{3}\,(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}), \\
q^{[\lambda+1]}_{r+1,s+1} &= q^{[\lambda]}_{r+1,s+1} + \frac{h}{3}\,(f^{[\lambda]}_{r+1,s+1} - f^{[\lambda-1]}_{r+1,s+1}), \\
f^{[\lambda+1]}_{r+1,s+1} &= f(x_{r+1}, y_{s+1}, z^{[\lambda+1]}_{r+1,s+1}, p^{[\lambda+1]}_{r+1,s+1}, q^{[\lambda+1]}_{r+1,s+1}).
\end{aligned}
\tag{30}
$$

Für die exakte Lösung der Differentialgleichung gelten entsprechende Gleichungen (vgl. (27)) mit dem Kubaturfehler $S^{**}_{3,3}$ und den Quadraturfehlern $\bar{S}^{**}_3$ und $\bar{\bar{S}}^{**}_3$, die wie folgt abgeschätzt werden können:

$$
\begin{aligned}
|S^{**}_{3,3}| &\leqq \frac{h\,k^5}{45}\left|\frac{\partial^4}{\partial x^4}F\right|_{\mathrm{Max}} + \frac{k\,h^5}{45}\left|\frac{\partial^4}{\partial y^4}F\right|_{\mathrm{Max}} + \frac{h^5 k^5}{8100}\left|\frac{\partial^8}{\partial x^4\partial y^4}F\right|_{\mathrm{Max}}, \\
|\bar{S}^{**}_3| &\leqq \frac{k^5}{90}\left|\frac{\partial^4}{\partial y^4}F\right|_{\mathrm{Max}}, \qquad |\bar{\bar{S}}^{**}_3| \leqq \frac{h^5}{90}\left|\frac{\partial^4}{\partial x^4}F\right|_{\mathrm{Max}}.
\end{aligned}
\tag{31}
$$

6. Konvergenz der Anfangsiteration

Berücksichtigt man, daß die Differentialgleichung (1) eine Lipschitz-Bedingung mit der Lipschitz-Konstanten M erfüllt und setzt man

$$z_{\varrho,\sigma}^{[\lambda+1]} - z_{\varrho,\sigma}^{[\lambda]} = \delta_{\varrho,\sigma}^{[\lambda]}, \qquad p_{\varrho,\sigma}^{[\lambda+1]} - p_{\varrho,\sigma}^{[\lambda]} = \bar{\delta}_{\varrho,\sigma}^{[\lambda]}, \qquad q_{\varrho,\sigma}^{[\lambda+1]} - q_{\varrho,\sigma}^{[\lambda]} = \bar{\bar{\delta}}_{\varrho,\sigma}^{[\lambda]}, \tag{32}$$

ferner

$$|\delta_{\varrho,\sigma}^{[\lambda]}| + |\bar{\delta}_{\varrho,\sigma}^{[\lambda]}| + |\bar{\bar{\delta}}_{\varrho,\sigma}^{[\lambda]}| = w_{\varrho,\sigma}^{[\lambda]},$$

so gelten für die Gleichungen (21) die Abschätzungen

$$\begin{aligned} |\delta_{r,s}^{[\lambda]}| &\leqq h\,k\,M \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |\gamma_{r,\mu}^{m}|\,|\gamma_{s,\nu}^{n}|\, w_{\mu,\nu}^{[\lambda-1]}, \\ |\bar{\delta}_{r,s}^{[\lambda]}| &\leqq k\,M \sum_{\nu=1}^{n} |\gamma_{s,\nu}^{n}|\, w_{r,\nu}^{[\lambda-1]}, \\ |\bar{\bar{\delta}}_{r,s}^{[\lambda]}| &\leqq h\,M \sum_{\mu=1}^{m} |\gamma_{r,\mu}^{m}|\, w_{\mu,s}^{[\lambda-1]}, \\ (r = 1, 2, &\ldots, m;\ s = 1, 2, \ldots, n). \end{aligned} \tag{33}$$

Die Addition dieser Gleichungen liefert

$$w_{r,s}^{[\lambda]} \leqq h\,k\,M \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |\gamma_{r,\mu}^{m}|\,|\gamma_{s,\nu}^{n}|\, w_{\mu,\nu}^{[\lambda-1]} + k\,M \sum_{\nu=1}^{n} |\gamma_{s,\nu}^{n}|\, w_{r,\nu}^{[\lambda-1]} + h\,M \sum_{\mu=1}^{m} |\gamma_{r,\mu}^{m}|\, w_{\mu,s}^{[\lambda-1]}. \tag{34}$$

Dieses Ungleichungssystem hat die Form

$$w_{r,s}^{[\lambda]} \leqq \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |A_{\mu,\nu}^{r,s}|\, w_{\mu,\nu}^{[\lambda-1]}. \tag{35}$$

Sind alle Eigenwerte der Matrix

$$\mathfrak{A} = \left(|A_{\mu,\nu}^{r,s}|\right)$$

dem Betrage nach <1, so ist die Konvergenz der Reihen

$$\sum_{\lambda=0}^{\infty} \delta_{r,s}^{[\lambda]}, \qquad \sum_{\lambda=0}^{\infty} \bar{\delta}_{r,s}^{[\lambda]}, \qquad \sum_{\lambda=0}^{\infty} \bar{\bar{\delta}}_{r,s}^{[\lambda]}, \qquad (r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n),$$

gesichert. Daraus ergibt sich die Forderung

$$\operatorname*{Max}_{r,s} \left\{ \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |A_{\mu,\nu}^{r,s}| \right\} < 1, \qquad \text{(Zeilensummenkriterium)},$$

oder

$$\operatorname*{Max}_{\mu,\nu} \left\{ \sum_{r=1}^{m} \sum_{s=1}^{n} |A_{\mu,\nu}^{r,s}| \right\} < 1, \qquad \text{(Spaltensummenkriterium)}. \tag{36}$$

Hieraus erhält man obere Schranken für die Schrittweiten h und k. Für $k = h$ $m = n = 2$ lauten diese Bedingungen

$$h < \frac{29}{30}\left(\sqrt{1 + \frac{720}{842 \cdot M}} - 1\right),$$

bzw.

$$h < \frac{1}{2}\left(\sqrt{1 + \frac{1}{M}} - 1\right). \tag{37}$$

7. Konvergenz der Iterationen für die fortlaufende Rechnung

Mit den Bezeichnungen (32) gilt für die Gleichungen (26) die Abschätzung:

$$w^{[\lambda]}_{r+1,s+1} \leqq \left(h\,k\,|\alpha^*_{m,0}|\,|\alpha^*_{n,0}| + k\,|\alpha^*_{n,0}| + h\,|\alpha^*_{m,0}|\right) M\, w^{[\lambda-1]}_{r+1,s+1}.$$

Die Konvergenz der Reihen

$$\sum_{\lambda=0}^{\infty} \delta^{[\lambda]}_{r+1,s+1}, \quad \sum_{\lambda=0}^{\infty} \bar{\delta}^{[\lambda]}_{r+1,s+1}, \quad \sum_{\lambda=0}^{\infty} \bar{\bar{\delta}}^{[\lambda]}_{r+1,s+1}$$

ist daher gesichert, wenn

$$M\left(h\,k\,|\alpha^*_{m,0}|\,|\alpha^*_{n,0}| + k\,|\alpha^*_{n,0}| + h\,|\alpha^*_{m,0}|\right) < 1. \tag{38}$$

Mit $k=h$ und $n=m$ erhält man:

$$h < \frac{1}{|\alpha^*_{m,0}|}\left(\sqrt{1+\frac{1}{M}} - 1\right).$$

Zahlenmäßig lautet diese Bedingung:

$$\begin{aligned}
h &< 2\left(\sqrt{1+\frac{1}{M}}-1\right) && \text{für} \quad m=1,\\
h &< \frac{12}{5}\left(\sqrt{1+\frac{1}{M}}-1\right) && \text{für} \quad m=2,\\
h &< \frac{24}{9}\left(\sqrt{1+\frac{1}{M}}-1\right) && \text{für} \quad m=3,\\
h &< \frac{720}{251}\left(\sqrt{1+\frac{1}{M}}-1\right) && \text{für} \quad m=4,\\
h &< \frac{288}{95}\left(\sqrt{1+\frac{1}{M}}-1\right) && \text{für} \quad m=5.
\end{aligned} \tag{39}$$

Für das Verfahren (30) lautet die Konvergenzbedingung:

$$h < 3\left(\sqrt{1+\frac{1}{M}} - 1\right). \tag{40}$$

8. Fehlerabschätzung für die Werte des Anfangsfeldes

Es werde vorausgesetzt, daß die Iterationen (21) bis zum Stillstand durchgeführt wurden, so daß die Indizes $[\lambda]$ fortgelassen werden können. Die Abrundungsfehler sollen außer acht gelassen werden.

Es wird angenommen, daß die Berechnung des Anfangsfeldes entlang der Geraden $y=y_0$ bis zu einer Stelle $x=x_\alpha$ fortgeschritten ist. Es sind daher bereits Werte $z_{\mu,\nu}$, $p_{\mu,\nu}$, $q_{\mu,\nu}$, $f_{\mu,\nu}$, ($\mu=0, 1, \ldots, \alpha$; $\nu=0, 1, \ldots, n$), bekannt. Denkt man sich die Anfangswerte auf den Geraden $x=x_\alpha$, $y=y_0$ vorgegeben, so lauten die Formeln (16) zur Berechnung von Näherungswerten $z_{\alpha+r,s}$, $p_{\alpha+r,s}$, $q_{\alpha+r,s}$, $f_{\alpha+r,s}$, ($\alpha=0, m, 2m, \ldots$):

$$\begin{aligned}
z_{\alpha+r,s} &= z_{\alpha+r,0} + z_{\alpha,s} - z_{\alpha,0} + h\,k \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} \gamma^m_{r,\mu}\gamma^n_{s,\nu} f_{\alpha+\mu,\nu},\\
p_{\alpha+r,s} &= p_{\alpha+r,0} + k \sum_{\nu=0}^{n} \gamma^n_{s,\nu} f_{\alpha+r,\nu},\\
q_{\alpha+r,s} &= q_{\alpha,s} + h \sum_{\mu=0}^{m} \gamma^m_{r,\mu} f_{\alpha+\mu,s},\\
f_{\alpha+r,s} &= f(x_{\alpha+r}, y_s, z_{\alpha+r,s}, p_{\alpha+r,s}, q_{\alpha+r,s}), \quad (r=1,2,\ldots,m;\ s=1,2,\ldots,n).
\end{aligned} \tag{41}$$

Dabei sind alle Werte auf der Geraden $y = y_0$ exakt vorgegeben, während die Werte auf $x = x_\alpha$ bereits durch dasselbe Verfahren berechnet wurden und daher mit einem Fehler behaftet sind.

Für die exakte Lösung der Differentialgleichung gilt:

$$\begin{aligned} z(x_{\alpha+r}, y_s) &= z(x_{\alpha+r}, y_0) + z(x_\alpha, y_s) - z(x_\alpha, y_0) + \\ &\qquad + h\,k \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \gamma_{r,\mu}^m \gamma_{s,\nu}^n F(x_{\alpha+\mu}, y_\nu) + S_{m+1,n+1}^{r,s}, \\ p(x_{\alpha+r}, y_s) &= p(x_{\alpha+r}, y_0) + k \sum_{\nu=0}^{n} \gamma_{s,\nu}^n F(x_{\alpha+r}, y_\nu) + \overline{S}_{n+1}^{r,s}, \\ q(x_{\alpha+r}, y_s) &= q(x_\alpha, y_s) + h \sum_{\mu=0}^{m} \gamma_{r,\mu}^m F(x_{\alpha+\mu}, y_s) + \overline{\overline{S}}_{m+1}^{r,s}. \end{aligned} \tag{42}$$

Setzt man für $l \geqq 1$

$$z_{j,l} - z(x_j, y_l) = \varepsilon_{j,l}, \qquad p_{j,l} - p(x_j, y_l) = \overline{\varepsilon}_{j,l}, \qquad q_{j,l} - q(x_j, y_l) = \overline{\overline{\varepsilon}}_{j,l},$$

subtrahiert die Gleichungen (42) von den entsprechenden Gleichungen (41), so erhält man mit der Lipschitz-Bedingung die Abschätzungen:

$$\begin{aligned} &|\varepsilon_{\alpha+r,s}| - h\,k\,M \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |\gamma_{r,\mu}^m|\,|\gamma_{s,\nu}^n| \left\{|\varepsilon_{\alpha+\mu,\nu}| + |\overline{\varepsilon}_{\alpha+\mu,\nu}| + |\overline{\overline{\varepsilon}}_{\alpha+\mu,\nu}|\right\} \leqq |R_{\alpha+r,s}|, \\ &|\overline{\varepsilon}_{\alpha+r,s}| - k\,M \sum_{\nu=1}^{n} |\gamma_{s,\nu}^n| \left\{|\varepsilon_{\alpha+r,\nu}| + |\overline{\varepsilon}_{\alpha+r,\nu}| + |\overline{\overline{\varepsilon}}_{\alpha+r,\nu}|\right\} \leqq |\overline{R}_{\alpha+r,s}|, \\ &|\overline{\overline{\varepsilon}}_{\alpha+r,s}| - h\,M \sum_{\mu=1}^{m} |\gamma_{r,\mu}^m| \left\{|\varepsilon_{\alpha+\mu,s}| + |\overline{\varepsilon}_{\alpha+\mu,s}| + |\overline{\overline{\varepsilon}}_{\alpha+\mu,s}|\right\} \leqq |\overline{\overline{R}}_{\alpha+r,s}|, \end{aligned} \tag{43}$$

mit

$$\begin{aligned} |R_{\alpha+r,s}| &= |\varepsilon_{\alpha,s}| + h \cdot k\,M \sum_{\nu=1}^{n} |\gamma_{r,0}^m|\,|\gamma_{s,\nu}^n| \left\{|\varepsilon_{\alpha,\nu}| + |\overline{\varepsilon}_{\alpha,\nu}| + |\overline{\overline{\varepsilon}}_{\alpha,\nu}|\right\} + |S_{m+1,n+1}^{r,s}|, \\ |\overline{R}_{\alpha+r,s}| &= |\overline{S}_{n+1}^{r,s}|, \\ |\overline{\overline{R}}_{\alpha+r,s}| &= |\overline{\overline{\varepsilon}}_{\alpha,s}| + h\,M\,|\gamma_{r,0}^m| \left\{|\varepsilon_{\alpha,s}| + |\overline{\varepsilon}_{\alpha,s}| + |\overline{\overline{\varepsilon}}_{\alpha,s}|\right\} + |\overline{\overline{S}}_{m+1}^{r,s}|, \\ &(r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n). \end{aligned} \tag{44}$$

Die rechte Seite dieses linearen inhomogenen Ungleichungssystems ist bekannt. Die Matrix des Systems, hier kurz mit $\mathfrak{D} = (d_{ik})$ bezeichnet, ist von „monotoner Art" [5], denn das hinreichende Kriterium hierfür ist erfüllt:

1. Es ist $d_{ik} \leqq 0$ für $i \neq k$.
2. $\mathfrak{D}$ zerfällt nicht.
3. Es existieren $\mathfrak{t} > 0$ und $\mathfrak{z} > 0$ mit $\mathfrak{D}\mathfrak{z} = \mathfrak{t}$.

Die 1. Bedingung ist immer erfüllbar bei geeigneter Reihenfolge der Summation, 2. ist stets erfüllt. Um zu zeigen, daß auch die 3. Bedingung erfüllt ist, werden die Ungleichungen (43) majorisiert:

$$\begin{aligned} &Z_{\alpha+r,s} - h\,k\,M \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} |\gamma_{r,\mu}^m|\,|\gamma_{s,\nu}^n|\,(Z_{\alpha+\mu,\nu} + \overline{Z}_{\alpha+\mu,\nu} + \overline{\overline{Z}}_{\alpha+\mu,\nu}) = T_{\alpha+r,s}, \\ &\overline{Z}_{\alpha+r,s} - k\,M \sum_{\nu=1}^{n} |\gamma_{s,\nu}^n|\,(Z_{\alpha+r,\nu} + \overline{Z}_{\alpha+r,\nu} + \overline{\overline{Z}}_{\alpha+r,\nu}) = \overline{T}_{\alpha+r,s}, \\ &\overline{\overline{Z}}_{\alpha+r,s} - h\,M \sum_{\mu=1}^{m} |\gamma_{r,\mu}^m|\,(Z_{\alpha+\mu,s} + \overline{Z}_{\alpha+\mu,s} + \overline{\overline{Z}}_{\alpha+\mu,s}) = \overline{\overline{T}}_{\alpha+r,s}, \\ &\qquad (r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n), \end{aligned} \tag{45}$$

mit

$$\begin{pmatrix} |\varepsilon_{\alpha+\mu,\nu}| \\ |\bar{\varepsilon}_{\alpha+\mu,\nu}| \\ |\bar{\bar{\varepsilon}}_{\alpha+\mu,\nu}| \end{pmatrix} \leqq \begin{pmatrix} Z_{\alpha+\mu,\nu} \\ \bar{Z}_{\alpha+\mu,\nu} \\ \bar{\bar{Z}}_{\alpha+\mu,\nu} \end{pmatrix}, \qquad \begin{pmatrix} |R_{\alpha+r,s}| \\ |\bar{R}_{\alpha+r,s}| \\ |\bar{\bar{R}}_{\alpha+r,s}| \end{pmatrix} \leqq \begin{pmatrix} T_{\alpha+r,s} \\ \bar{T}_{\alpha+r,s} \\ \bar{\bar{T}}_{\alpha+r,s} \end{pmatrix}, \tag{46}$$

$$(\mu, r = 1, 2, \ldots, m; \; \nu, s = 1, 2, \ldots, n).$$

Das Gleichungssystem (45) hat mit $\mathfrak{E}$ als Einheitsmatrix die Form

$$\mathfrak{z} - (\mathfrak{E} - \mathfrak{D})\,\mathfrak{z} = \mathfrak{t} \tag{47}$$

und mit $\mathfrak{E} - \mathfrak{D} = \mathfrak{A}$ die Lösung

$$\mathfrak{z} = (\mathfrak{E} + \mathfrak{A} + \mathfrak{A}^2 + \cdots)\,\mathfrak{t}. \tag{48}$$

Diese Matrizenreihe konvergiert, wenn alle Eigenwerte der Matrix $\mathfrak{A}$ dem Betrag nach <1 sind. Nach (36) ist das stets der Fall. Damit haben wir eine Darstellung der Form

$$\mathfrak{z} = \mathfrak{G}\,\mathfrak{t}.$$

Die Matrix $\mathfrak{G}$ hat nur positive Elemente. Wegen $\mathfrak{t} > 0$ (d.h. jedes Element von $\mathfrak{t}$ ist >0) ist auch $\mathfrak{z} > 0$ und damit auch die 3. Bedingung erfüllt.

Wegen der Monotonie des Systems (45) sind dann die $Z_{\varkappa,\lambda}$, $\bar{Z}_{\varkappa,\lambda}$, $\bar{\bar{Z}}_{\varkappa,\lambda}$ obere Schranken der $|\varepsilon_{\varkappa,\lambda}|$, $|\bar{\varepsilon}_{\varkappa,\lambda}|$, $|\bar{\bar{\varepsilon}}_{\varkappa,\lambda}|$.

Sei $a = \operatorname{Max}(\mathfrak{A})$ der Betrag des größten Elementes der Matrix $\mathfrak{A}$, so gilt für den Betrag des größten Elementes $\operatorname{Max}(\mathfrak{A}^2)$ der Matrix $\mathfrak{A}^2$:

$$\operatorname{Max}(\mathfrak{A}^2) \leqq \Bigl(\operatorname*{Max}_i \sum_k a_{ik}\Bigr) \operatorname*{Max}_{i,k} a_{ik} = A\,a,$$

allgemein

$$\operatorname{Max}(\mathfrak{A}^n) \leqq A^{n-1} a.$$

Nach (48) erhält man so für $A < 1$ die Fehlerabschätzung:

$$\begin{aligned} |\varepsilon_{\alpha+r,s}| &\leqq T_{\alpha+r,s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T_{\alpha+\mu,\nu} + \bar{T}_{\alpha+\mu,\nu} + \bar{\bar{T}}_{\alpha+\mu,\nu}), \\ |\bar{\varepsilon}_{\alpha+r,s}| &\leqq \bar{T}_{\alpha+r,s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T_{\alpha+\mu,\nu} + \bar{T}_{\alpha+\mu,\nu} + \bar{\bar{T}}_{\alpha+\mu,\nu}), \\ |\bar{\bar{\varepsilon}}_{\alpha+r,s}| &\leqq \bar{\bar{T}}_{\alpha+r,s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T_{\alpha+\mu,\nu} + \bar{T}_{\alpha+\mu,\nu} + \bar{\bar{T}}_{\alpha+\mu,\nu}). \end{aligned} \tag{49}$$

Für $\alpha = 0$ treten in diesen Ungleichungen anstelle der $T_{\varkappa,\lambda}$, $\bar{T}_{\varkappa,\lambda}$, $\bar{\bar{T}}_{\varkappa,\lambda}$ die oberen Schranken (19) für die reinen Kubatur- und Quadraturfehler.

Sei jetzt die Berechnung des Anfangsfeldes entlang der Geraden $x = x_0$ bis zur Stelle $y = y_\beta$ fortgeschritten, $(\beta = n, 2n, 3n, \ldots)$. Man erhält dann ganz ähnlich für die Fehlerbeträge der Näherungen $z_{r,\beta+s}$, $p_{r,\beta+s}$, $q_{r,\beta+s}$ Abschätzungen, die nach Betrachtung der Abschätzungen (49) gleich hingeschrieben werden

können:

$$
\begin{aligned}
|\varepsilon_{r,\beta+s}| &\leqq T^*_{r,\beta+s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T^*_{\mu,\beta+\nu} + \overline{T}^*_{\mu,\beta+\nu} + \overline{\overline{T}}^*_{\mu,\beta+\nu}), \\
|\overline{\varepsilon}_{r,\beta+s}| &\leqq \overline{T}^*_{r,\beta+s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T^*_{\mu,\beta+\nu} + \overline{T}^*_{\mu,\beta+\nu} + \overline{\overline{T}}^*_{\mu,\beta+\nu}), \\
|\overline{\overline{\varepsilon}}_{r,\beta+s}| &\leqq \overline{\overline{T}}^*_{r,\beta+s} + \frac{a}{1-A} \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (T^*_{\mu,\beta+\nu} + \overline{T}^*_{\mu,\beta+\nu} + \overline{\overline{T}}^*_{\mu,\beta+\nu}),
\end{aligned}
\tag{50}
$$

mit

$$
\begin{aligned}
&T^*_{\mu,\beta+\nu} \geqq |\varepsilon_{\mu,\beta}| + h\,k\,M\,|\gamma^n_{s,0}| \sum_{\mu=1}^{m} |\gamma^m_{r,\mu}| \left\{|\varepsilon_{\mu,\beta}| + |\overline{\varepsilon}_{\mu,\beta}| + |\overline{\overline{\varepsilon}}_{\mu,\beta}|\right\} + |S^{r,s}_{m+1,n+1}|, \\
&\overline{T}^*_{\mu,\beta+\nu} \geqq |\overline{\varepsilon}_{\mu,\beta}| + k\,M\,|\gamma^n_{s,0}| \left\{|\varepsilon_{r,\beta}| + |\overline{\varepsilon}_{r,\beta}| + |\overline{\overline{\varepsilon}}_{r,\beta}|\right\} + |\overline{S}^{r,s}_{n+1}|, \\
&\overline{\overline{T}}^*_{\mu,\beta+\nu} \geqq |\overline{\overline{S}}^{r,s}_{m+1}|, \\
&\qquad (\beta = n, 2n, 3n, \ldots;\ r = 1, 2, \ldots, m;\ s = 1, 2, \ldots, n).
\end{aligned}
\tag{51}
$$

Nach (45) haben die Werte a und A für $k=h$ stets die Form:

$$
a = b\,M\,h,
$$
$$
A = B\,M\,h
$$

mit konstanten b und B, so daß

$$
\frac{a}{1-A} = \frac{b\,M\,h}{1-B\,M\,h} \quad \text{mit} \quad B\,M\,h < 1 \quad \text{ist.} \tag{52}
$$

Für alle Verfahren der fortlaufenden Rechnung ergeben sich wegen der Tatsache, daß die Fehler nur dem Betrag nach abgeschätzt werden können, sehr ungünstige Fehlerschranken, die nach wenigen Rechenschritten unbrauchbar werden.

9. Beispiel

Es wurde bewußt eine nichtlineare Differentialgleichung „konstruiert“, um die exakte Lösung mit den Werten der Näherungslösung vergleichen zu können.

Differentialgleichung:

$$
\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y} \cdot \frac{1}{z} + z
$$

oder:

$$
s = \frac{p \cdot q}{z} + z,
$$

Anfangsbedingungen:

$$
z(x,0) = e^{x^2}, \qquad z(0,y) = \cos y.
$$

Daraus ergibt sich:

$$
p(x,0) = 2x\,e^{x^2}, \qquad q(0,y) = -\sin y,
$$
$$
p(0,y) = y\cos y, \qquad q(x,0) = x\,e^{x^2}.
$$

Die beiden letzten Bedingungen werden bestimmt aus

$$
p'(0,y) = -p(0,y)\,\mathrm{tg}\,y + \cos y, \qquad p(0,0) = 0,
$$
$$
q'(x,0) = 2q(x,0)\,x + e^{x^2}, \qquad q(0,0) = 0.
$$

Tabelle 6. *Näherungen* $z_{\mu,\nu}$

ν	μ = 0	1	2	3	4	5	6	7	8	ν
8	0,696707	0,762269	0,850912	0,969050	1,125881	1,334486	1,613797	1,990985	2,505751	8
7	0,764842	0,828520	0,915653	1,032393	1,187559	1,393615	1,668478	2,037871	2,539382	7
6	0,825336	0,885160	0,968541	1,081142	1,231253	1,430456	1,695606	2,050452	2,529498	6
5	0,877583	0,931832	1,009458	1,115594	1,257871	1,446869	1,697952	2,032770	2,482879	5
4	0,921061	0,968272	1,038493	1,136280	1,268413	1,444453	1,678283	1,989317	2,405467	4
3	0,955336	0,994314	1,055820	1,143718	1,264050	1,425175	1,639380	1,923767	2,303213	3
2	0,980067	1,009908	1,061694	1,138667	1,245903	1,390724	1,583851	1,840217	2,181093	2
1	0,995004	1,015102	1,056533	1,121844	1,215309	1,343104	1,514372	1,741892	2,044157	1
0	1,000000	1,010050	1,040811	1,094174	1,173511	1,284025	1,433329	1,632316	1,896481	0

Fehler $\varepsilon_{\mu,\nu} \cdot 10^8$

ν	μ = 0	1	2	3	4	5	6	7	8	ν
8	0	−49	+3	−47	−49	−84	−31	+40	−52	8
7	0	−24	−29	−36	−17	−18	−4	−18	+17	7
6	0	−19	+1	−18	−3	−59	−1	+52	−35	6
5	0	−17	−2	32	+7	−20	+12	33	+7	5
4	0	−12	0	−13	−6	−58	0	+31	−65	4
3	0	10	+10	−27	+16	−20	+16	44	7	3
2	0	−5	+1	−7	−5	−57	−9	+34	−85	2
1	0	−2	+2	−20	+9	−11	+14	32	13	1
0	0	0	0	0	0	0	0	0	0	0

Lösung des Problems:

$$z = e^{x^2 + xy} \cdot \cos y .$$

Ergebnis der Rechnung:

Das Anfangsfeld (umrandet) wurde nach (11) und (20) bzw. (21) mit $m = n = 2$, $h = k = 0, 1$ berechnet. Für die fortlaufende Rechnung wurde Verfahren (29) bzw. (30) verwendet. Bei der Berechnung des Anfangsfeldes waren durchschnittlich drei, bei der fortlaufenden Rechnung nur ein bis zwei Iterationen erforderlich.

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(Eingegangen am 30. Oktober 1959)

Zur numerischen Behandlung von Anfangswertproblemen partieller hyperbolischer Differentialgleichungen zweiter Ordnung in zwei unabhängigen Veränderlichen

II. Das Cauchy-Problem *

W. TÖRNIG

Vorgelegt von W. TOLLMIEN

1. Einleitung

In diesem II. Teil der Arbeit werden numerische Verfahren zur Behandlung des Cauchy-Problems der Differentialgleichung

$$\frac{\partial^2 z}{\partial x\,\partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

mitgeteilt. Die Anfangswerte sind dabei auf einer streng monotonen Kurve vorgegeben. Nach einer passenden Koordinatentransformation kann ähnlich wie beim charakteristischen Anfangswertproblem vorgegangen werden. [Siehe Teil I.] In den Nummern 2. bis 5. werden Extrapolations- und Interpolationsverfahren zur Berechnung des Anfangsfeldes und für die fortlaufende Rechnung beschrieben und in 6. bis 9. theoretisch untersucht. Es gelingt, im Prinzip Fehlerabschätzungen für die Interpolationsverfahren anzugeben. In 10. wird für das Interpolationsverfahren der fortlaufenden Rechnung die Konvergenz der Näherungswerte gegen die Werte der exakten Lösung bei kleiner werdender Schrittweite gezeigt. In 11. wird ein Beispiel durchgerechnet. Die Frage nach der numerischen Stabilität der Verfahren wird an dieser Stelle nicht erörtert. (Siehe Teil I der Arbeit.) Der Teil I dieser Arbeit wird als bekannt vorausgesetzt, die Bezeichnungen werden teilweise übernommen. Wir können uns daher etwas kürzer fassen.

2. Ausgangsgleichungen

Vorgelegt sei das Anfangswertproblem (Cauchy-Problem):

$$\begin{gathered} s = f(x, y, z, p, q)\,, \\ z\big(x, \varphi(x)\big) = \alpha(x)\,, \quad p\big(x, \varphi(x)\big) = \beta(x)\,, \quad q\big(x, \varphi(x)\big) = \gamma(x) \end{gathered} \tag{1}$$

auf der Anfangskurve $y = \varphi(x)$.

* 2. Teil einer gekürzten Fassung der 1958 von der Fakultät für Natur- und Geisteswissenschaften der Bergakademie Clausthal angenommenen Dissertation des Verfassers; Referenten: Prof. Dr. H. KÖNIG und Prof. Dr. H. MENZEL.

Die Frage nach der Existenz und Eindeutigkeit der Lösung dieses Problems beantwortet der [2], [3]

Satz 1. *In dem Rechteck*

$$\mathfrak{R}:\quad \alpha_1 < x < \beta_1,\quad \gamma_1 < y < \delta_1,\quad z, u, v \quad \textit{beliebig},$$

sei die Funktion $f(x, y, z, u, v)$ stetig, in jedem abgeschlossenen Teilgebiet g von $\mathfrak{R}$ beschränkt und erfülle dort eine Lipschitz-Bedingung:

$$|f(x, y, z_2, u_2, v_2) - f(x, y, z_1, u_1, v_1)| \leqq M(|z_2 - z_1| + |u_2 - u_1| + |v_2 - v_1|)\,.$$

Es sei

$\mathfrak{K}$: $y = \varphi(x)$ mit $x_1 < x < x_2$ eine Kurve, die ganz in $\mathfrak{R}$ verläuft und die stetig differenzierbar und streng monoton ist. Dann hat das Problem (1) *in $\mathfrak{R}$ das eindeutig bestimmte Integral $z = \psi(x, y)$ mit $\psi\big(x, \varphi(x)\big) = \alpha(x)$.*

Die p und q können auf der Anfangskurve beliebig vorgegeben werden, müssen jedoch der Streifenbedingung

$$dz = p\,dx + q\,dy \tag{2}$$

genügen.

Das Problem (1) wird modifiziert, indem die neuen Koordinaten

$$\xi = x, \qquad \eta = [\varphi(y)]^{(-1)}, \quad \text{also} \quad x = \xi, \qquad y = \varphi(\eta)$$

eingeführt werden; es lautet dann:

$$z_{\xi,\eta} = G(\xi, \eta, z, z_\xi, z_\eta)$$

mit den Anfangswerten $z(\xi, \xi)$, $p(\xi, \xi)$, $q(\xi, \xi)$ auf der Anfangsgeraden $\eta = \xi$.

Wir schreiben im folgenden statt ξ, η wieder x, y, statt G wieder f, so daß also jetzt das Problem

$$\begin{gathered} s = f(x, y, z, p, q)\,, \\ z(x, x) = \alpha(x)\,, \qquad p(x, x) = \beta(x)\,, \qquad q(x, x) = \gamma(x) \end{gathered} \tag{3}$$

zur Lösung vorliegt.

Für die Entwicklung der numerischen Verfahren ist es nützlich, die Gebiete $y \geqq x$ und $y \leqq x$ gesondert zu betrachten. Dabei soll die Gerade $y = x$ jeweils zu beiden Gebieten gehören. Es werden daher „Formeln für Werte oberhalb $y = x$" und „Formeln für Werte unterhalb $y = x$" entwickelt.

Sei (ξ, η) ein Punkt oberhalb $y = x$, der zum Definitionsbereich des Problems (3) gehört, so ist dieses unter den Voraussetzungen des Satz 1 äquivalent den Integralformen:

$$\begin{aligned} z(\xi, \eta) &= z(\xi, \xi) + \int_\xi^\eta q(x, x)\,dx + \int_\xi^\eta \int_\eta^x f(x, y, z, p, q)\,dy\,dx\,, \\ p(\xi, \eta) &= p(\xi, \xi) + \int_\xi^\eta f(\xi, y, z, p, q)\,dy\,, \\ q(\xi, \eta) &= q(\eta, \eta) - \int_\xi^\eta f(x, \eta, z, p, q)\,dx\,. \end{aligned} \tag{4}$$

Ist (ξ, η) ein Punkt unterhalb $y=x$, so wird entsprechend:

$$
\begin{aligned}
z(\eta, \xi) &= z(\eta, \eta) - \int_{\xi}^{\eta} q(x, x)\, dx - \int_{\xi}^{\eta} \int_{\xi}^{x} f(x, y, z, p, q)\, dy\, dx, \\
p(\eta, \xi) &= p(\eta, \eta) - \int_{\xi}^{\eta} f(\xi, y, z, p, q)\, dy, \\
q(\eta, \xi) &= q(\xi, \xi) + \int_{\xi}^{\eta} f(x, \eta, z, p, q)\, dx.
\end{aligned} \tag{5}
$$

Mit Hilfe der Streifenbedingung (2) läßt sich das Integral

$$\int_{\xi}^{\eta} q(x, x)\, dx \quad \text{durch} \quad -\int_{\xi}^{\eta} p(x, x)\, dx + z(\eta, \eta) - z(\xi, \xi) \tag{6}$$

ersetzen und man erhält gleichwertige Vorschriften. Es wird darauf aber nicht weiter eingegangen.

Die Intervalle $\alpha_1 < \alpha \leqq x \leqq \beta < \beta_1$, $\gamma_1 < \gamma \leqq y \leqq \delta < \delta_1$ werden unterteilt durch

$$
\begin{aligned}
\alpha &= x_0 < x_1 < \cdots < x_j = \beta, \\
\beta &= y_0 < y_1 < \cdots < y_l = \delta.
\end{aligned} \tag{7}
$$

Es sei ferner stets

$$x_{\mu+1} - x_{\mu} = y_{\nu+1} - y_{\nu} = h, \qquad (\mu = 0, 1, 2, \ldots;\ \nu = 0, 1, 2, \ldots).$$

3. Extrapolationsverfahren

Die Rechnung sei bereits so weit fortgeschritten, daß Näherungswerte

$$z_{r+\mu, s-\nu}, p_{r+\mu, s-\nu}, q_{r+\mu, s-\nu}, f_{r+\mu, s-\nu}, \quad (r, s \geqq 0;\ \mu = 0, 1, \ldots, m;\ \nu = 0, 1, \ldots, m-\mu),$$

bekannt sind. Es sei bemerkt, daß die im folgenden verwendeten Approximationen nicht die einzig möglichen sind. Da durch Extrapolation aber hauptsächlich Rohwerte gefunden werden sollen, dürfen wir uns beschränken.

Es wird dann mit $\xi = x_r$, $\eta = y_{s+1}$ in (4) $f(x, y, z, p, q)$ durch das Polynom ersetzt, das an den Stellen $(x_{r+\mu}, y_{s-\nu})$, $\mu = 0, 1, \ldots, m$; $\nu = 0, 1, \ldots, m-\mu$) die Werte $f_{r+\mu, s-\nu}$ annimmt [*1*], [*4*]:

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{m-\mu} \frac{U^{(\mu)}}{\mu!} \frac{V^{(-\nu)}}{\nu!} \Delta_x^{\mu} \nabla_y^{\nu} f_{r,s}, \tag{8}$$

mit

$$x - x_r = uh, \qquad y - y_s = vh. \tag{9}$$

Entsprechend wird ersetzt:

$$
\begin{aligned}
&f(x_r, y, z, p, q) &&\text{durch} \quad \sum_{\nu=0}^{m} \frac{V^{(-\nu)}}{\nu!} \nabla_y^{\nu} f_{r,s}, \\
&f(x, y_{s+1}, z, p, q) &&\text{durch} \quad \sum_{\mu=0}^{m} \frac{U^{(\mu+1)-1}}{\mu!} \Delta_x^{\mu} f_{r+1, s+1}, \\
&q\big(x, x + (y_s - x_r)\big) &&\text{durch} \quad \sum_{\mu=0}^{m+1} \frac{U^{(\mu)}}{\mu!} \Delta_x^{\mu} q_r.
\end{aligned} \tag{10}
$$

Dabei ist gesetzt:

$$U^{(\mu)} = u(u-1)(u-2)\dots(u-\mu+1),$$
$$V^{(-\nu)} = v(v+1)(v+2)\dots(v+\nu-1), \tag{11}$$
$$U^{(\mu+1)-1} = \frac{U^{(\mu+1)}}{u}.$$

(Bei der Wahl $\xi = x_r$, $\eta = y_{s+1}$ ist die Anfangsgerade $y = x + (y_s - x_r)$.)

Führt man die Integrationen in (4) aus und berücksichtigt, daß $y = x + (y_s - x_r)$ in $v = u$ übergeht, so erhält man für Werte oberhalb $y = x$:

$$\begin{aligned}
z_{r,s+1} &= z_{r,s} + h \sum_{\mu=0}^{m+1} (-1)^\mu \beta^*_\mu \Delta^\mu_x q_r - h^2 \sum_{\mu=0}^{m} \sum_{\nu=0}^{m-\mu} \beta_{\mu,\nu} \Delta^\mu_x \nabla^\nu_y f_{r,s},\\
p_{r,s-1} &= p_{r,s} + h \sum_{\nu=0}^{m} \beta_\nu \nabla^\nu_y f_{r,s},\\
q_{r,s+1} &= q_{r+1,s+1} - h \sum_{\mu=0}^{m} (-1)^\mu \beta_\mu \Delta^\mu_x f_{r+1,s+1},\\
f_{r,s+1} &= f(x_r, y_{s+1}, z_{r,s+1}, p_{r,s+1}, q_{r,s+1}),
\end{aligned} \tag{12}$$

mit

$$\begin{aligned}
\beta_{\mu,\nu} &= \frac{(-1)^{\mu+\nu}}{\mu!\,\nu!} \int_0^1 (u-1)\,u\,(u+1)\dots(u+\mu-2) \left[\int_0^u (v-1)(v-2)\dots(v-\nu)\,dv\right] du,\\
\beta_\lambda &= \frac{1}{\lambda!}\int_0^1 u(u+1)\dots(u+\lambda-1)\,du, \quad \beta^*_\lambda = \frac{1}{\lambda!}\int_0^1 (u-1)\,u\,(u+1)\dots(u+\lambda-2)\,du.
\end{aligned} \tag{13}$$

Tabelle 1. *Gewichte* $\beta_{\mu,\nu}$:

$\beta_{0,0} = \frac{12}{24}$	$\beta_{1,0} = \frac{4}{24}$	$\beta_{2,0} = -\frac{1}{24}$
$\beta_{0,1} = \frac{8}{24}$	$\beta_{1,1} = \frac{2}{24}$	
$\beta_{0,2} = \frac{5}{24}$		

(Für die Gewichte β_λ und β^*_λ siehe Tabellen 1 und 4 in Teil I.)

Die Abschätzung der Kubatur- und Quadraturfehler wird bei allen Verfahren zur Lösung des Cauchy-Problems recht umständlich. Da man bei der Extrapolation diesen Fehler wohl kaum berücksichtigen wird, sei hier auf eine Abschätzung verzichtet.

Es empfiehlt sich, die in (12) auftretenden Differenzen durch Funktionswerte zu ersetzen. Nach einiger Rechnung erhält man:

$$\begin{aligned}
z_{r,s+1} &= z_{r,s} + h \sum_{\mu=0}^{m+1} \alpha^*_{m+1,\mu} q_{r+\mu,s+\mu} - h^2 \sum_{\mu=0}^{m} \sum_{\nu=0}^{m-\mu} \alpha^m_{\mu,\nu} f_{r+\mu,s-\nu},\\
p_{r,s+1} &= p_{r,s} + h \sum_{\nu=0}^{m} \alpha_{m,\nu} f_{r,s-\nu},\\
q_{r,s+1} &= q_{r+1,s+1} - h \sum_{\mu=0}^{m} \alpha_{m,\mu} f_{r+1+\mu,s+1},\\
f_{r,s+1} &= f(x_r, y_{s+1}, z_{r,s+1}, p_{r,s+1}, q_{r,s+1}),
\end{aligned} \tag{14}$$

mit

$$\begin{aligned}
\alpha_{\mu,\nu}^{m} &= (-1)^{\mu+\nu} \sum_{j=\mu}^{m-\nu} \sum_{l=\nu}^{m-j} (-1)^{j} \beta_{j,l} \binom{j}{\mu} \binom{l}{\nu}, \\
\alpha_{m,\nu} &= (-1)^{\nu} \sum_{l=\nu}^{m} \binom{l}{\nu} \beta_{l}, \\
\alpha_{m+1,\mu}^{*} &= (-1)^{\mu} \sum_{l=\mu}^{m+1} \binom{l}{\mu} \beta_{l}^{*}.
\end{aligned} \tag{15}$$

Tabelle 2. *Gewichte* $\alpha_{\mu,\nu}^{m}$:

$m=0$:	$\alpha_{0,0}^{0}=\frac{1}{2}$		
$m=1$:	$\alpha_{0,0}^{1}=\frac{4}{6}$	$\alpha_{1,0}^{1}=\frac{1}{6}$	
	$\alpha_{0,1}^{1}=-\frac{2}{6}$		
$m=2$:	$\alpha_{0,0}^{2}=\frac{18}{24}$	$\alpha_{1,0}^{2}=\frac{8}{24}$	$\alpha_{2,0}^{2}=-\frac{1}{24}$
	$\alpha_{0,1}^{2}=-\frac{16}{24}$	$\alpha_{1,1}^{2}=-\frac{2}{24}$	
	$\alpha_{0,2}^{2}=\frac{5}{24}$		

(Zu den Gewichten $\alpha_{m,\nu}$ und $\alpha_{m+1,\mu}^{*}$ siehe Tabellen 2 und 5 in Teil I.)

Für die Entwicklung entsprechender Formeln für Werte unterhalb $y=x$ sei vorausgesetzt, daß die Rechnung schon weiter fortgeschritten ist und Näherungswerte $z_{r+1-\mu,s+1+\nu}$, $p_{r+1-\mu,s+1+\nu}$, $q_{r+1-\mu,s+1+\nu}$, $f_{r+1-\mu,s+1+\nu}$ bekannt sind, ($\mu=0, 1, \ldots, m$; $\nu=0, 1, \ldots, m-\mu$). Es ergeben sich dann die (14) ganz ähnlich gebauten Extrapolationsformeln für Werte unterhalb $y=x$:

$$\begin{aligned}
z_{r+1,s} &= z_{r+1,s+1} - h \sum_{\mu=0}^{m+1} \alpha_{m+1,\mu}^{*} q_{r+1-\mu,s+1-\mu} - h^{2} \sum_{\mu=0}^{m} \sum_{\nu=0}^{m-\mu} \alpha_{\mu,\nu}^{m} f_{r+1-\mu,s+1+\nu}, \\
p_{r+1,s} &= p_{r+1,s+1} - h \sum_{\nu=0}^{m} \alpha_{m,\nu} f_{r+1,s+1+\nu}, \\
q_{r+1,s} &= q_{r,s} + h \sum_{\mu=0}^{m} \alpha_{m,\mu} f_{r-\mu,s}, \\
f_{r+1,s} &= f(x_{r+1}, y_{s}, z_{r+1,s}, p_{r+1,s}, q_{r+1,s}).
\end{aligned} \tag{16}$$

4. Berechnung des Anfangsfeldes

Um die soeben in 3. und die in der nächsten Nummer 5. angegebenen Interpolationsformeln für die fortlaufende Rechnung anwenden zu können, benötigt man schon entlang der Geraden $y=x$ ein „Anfangsfeld" von Werten. Es soll hier speziell ein Interpolationsverfahren zur Berechnung dieses Anfangsfeldes angegeben werden.

Auf der Geraden $y=x$ sind die Anfangswerte $z(x_r, y_r)$, $p(x_r, y_r)$, $q(x_r, y_r)$, $F(x_r, y_r)$ vorgegeben. Man verschafft sich dann zuerst Rohwerte $z_{r,r+k}^{[0]}$, $p_{r,r+k}^{[0]}$, $q_{r,r+k}^{[0]}$, $f_{r,r+k}^{[0]}$, ($r=0, 1, \ldots, m$; $k=-r, -r+1, \ldots, -1, +1, \ldots, +m-r$) mit Hilfe der Formeln (14) und (16), die etwas umgeschrieben werden und dann lauten:

Für Werte oberhalb $y=x$, $(k\geqq 1)$:

$$\left.\begin{aligned}
z^{[0]}_{r,r+k} &= z^{[0]}_{r,r+k-1} + h\sum_{\mu=0}^{k}\alpha^{*}_{k,\mu}\, q^{[0]}_{r+\mu,r+k+\mu-1} - h^2\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1-\mu}\alpha^{k-1}_{\mu,\nu}\, f^{[0]}_{r+\mu,r+k-\nu-1},\\
p^{[0]}_{r,r+k} &= p^{[0]}_{r,r+k-1} + h\sum_{\nu=0}^{k-1}\alpha_{k-1,\nu}\, f^{[0]}_{r,r+k-\nu-1},\\
q^{[0]}_{r,r+k} &= q^{[0]}_{r+1,r+k} - h\sum_{\mu=0}^{k-1}\alpha_{k-1,\mu}\, f^{[0]}_{r+1+\mu,r+k},\\
f^{[0]}_{r,r+k} &= f\left(x_r, y_{r+k}, z^{[0]}_{r,r+k}, p^{[0]}_{r,r+k}, q^{[0]}_{r,r+k}\right).
\end{aligned}\right\}\quad (17)$$

Für Werte unterhalb $y=x$, $(k\leqq 1)$:

$$\left.\begin{aligned}
z^{[0]}_{r,r+k} &= z^{[0]}_{r,r+k+1} - h\sum_{\mu=0}^{k}\alpha^{*}_{k,\mu}\, q^{[0]}_{r-\mu,r+k-\mu+1} - h^2\sum_{\mu=0}^{k-1}\sum_{\nu=0}^{k-1-\mu}\alpha^{k-1}_{\mu,\nu}\, f^{[0]}_{r-\mu,r+k+1+\nu},\\
p^{[0]}_{r,r+k} &= p^{[0]}_{r,r+k+1} - h\sum_{\nu=0}^{k-1}\alpha_{k-1,\nu}\, f^{[0]}_{r,r+k+1+\nu},\\
q^{[0]}_{r,r+k} &= q^{[0]}_{r-1,r+k} + h\sum_{\mu=0}^{k-1}\alpha_{k-1,\mu}\, f^{[0]}_{r-\mu,r+k},\\
f^{[0]}_{r,r+k} &= f\left(x_r, y_{r+k}, z^{[0]}_{r,r+k}, p^{[0]}_{r,r+k}, q^{[0]}_{r,r+k}\right).
\end{aligned}\right\}\quad (18)$$

Dabei ist zu setzen:

$$z^{[0]}_{r,r} = z(x_r, y_r), \qquad p^{[0]}_{r,r} = p(x_r, y_r), \qquad q^{[0]}_{r,r} = q(x_r, y_r), \qquad f^{[0]}_{r,r} = F(x_r, y_r).$$

Die so gewonnenen Rohwerte werden nun iterativ verbessert:

Setzt man einmal in (4) $\xi = x_r$, $\eta = y_{r+k}$, ein andermal in (5) $\eta = x_r$, $\xi = y_{r+k}$, so erhält man — mit Berücksichtigung der Streifenbedingung — nach entsprechender Approximation der Integranden durch eindeutige Polynome folgende Interpolationsformeln, die als Iterationsvorschriften benutzt werden:

$$\left.\begin{aligned}
z^{[1]}_{r,r+k} &= z(x_r, y_r) + h\sum_{\mu=0}^{m}\eta^{r,k}_{\mu,m}\, q(x_\mu, y_\mu) + h^2\left[\sum_{\mu=0}^{m}\eta^{r,k}_{\mu,\mu,m} F(x_\mu, y_\mu) + \sum_{\substack{\mu=0\\ \mu\neq\nu}}^{m}\sum_{\nu=0}^{m}\eta^{r,k}_{\mu,\nu,m}\, f^{[0]}_{\mu,\nu}\right],\\
p^{[1]}_{r,r+k} &= p(x_r, y_r) + h\left[\eta^{r,k}_{r,m} F(x_r, y_r) + \sum_{\substack{\nu=0\\ \nu\neq r}}^{m}\eta^{r,k}_{\nu,m}\, f^{[0]}_{r,\nu}\right],\\
q^{[0]}_{r,r+k} &= q(x_{r+k}, y_{r+k}) - h\left[\eta^{r,k}_{r+k,m}\cdot F(x_{r+k}, y_{r+k}) + \sum_{\substack{\mu=0\\ \mu\neq r+k}}^{m}\eta^{r,k}_{\mu,m}\, f^{[0]}_{\mu,r+k}\right],\\
f^{[1]}_{r,r+k} &= f\left(x_r, y_{r+k}, z^{[1]}_{r,r+k}, p^{[1]}_{r,r+k}, q^{[1]}_{r,r+k}\right),
\end{aligned}\right\}\quad (19)$$

mit den Gewichten

$$\left.\begin{aligned}
\eta^{r,k}_{\mu,\nu,m} = \frac{(-1)^{\mu+\nu}}{\mu!\,\nu!\,(m-\mu)!\,(m-\nu)!}&\int_0^{k}(u+r)\dots(u+r-\mu+1)\times\\
&\times(u+r-\mu-1)\dots(u+r-m)\times\\
&\times\left[\int_k^{u}(v+r)\dots(v+r-\nu+1)(v+r-\nu-1)\dots(v+r-m)\,dv\right]du,\\
\eta^{r,k}_{\lambda,m} = \frac{(-1)^{m-\lambda}}{\lambda!\,(m-\lambda)!}&\int_0^{k}(u+r)\dots(u+r-\lambda+1)(u+r-\lambda-1)\dots(u+r-m)\,du.
\end{aligned}\right\}\quad (20)$$

Tabelle 3. *Gewichte* $\eta_{\lambda,2}^{r,k}$

r k λ	0 1	0 2	1 1	1 -1	2 -1	2 -2
0	$\frac{5}{12}$	$\frac{4}{12}$	$-\frac{1}{12}$	$-\frac{5}{12}$	$\frac{1}{12}$	$-\frac{4}{12}$
1	$\frac{8}{12}$	$\frac{16}{12}$	$\frac{8}{12}$	$-\frac{8}{12}$	$-\frac{8}{12}$	$\frac{16}{12}$
2	$-\frac{1}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{1}{12}$	$-\frac{5}{12}$	$-\frac{4}{12}$

Gewichte $\eta_{\mu,\nu,2}^{r,k}$

r k	0 1			0 2		
ν \ μ	0	1	2	0	1	2
2	$\frac{37}{1440}$	$\frac{28}{1440}$	$-\frac{5}{1440}$	$-\frac{176}{1440}$	$-\frac{704}{1440}$	$-\frac{80}{1440}$
1	$\frac{332}{1440}$	$-\frac{320}{1440}$	$\frac{52}{1440}$	$-\frac{704}{1440}$	$-\frac{1280}{1440}$	$\frac{64}{1440}$
0	$-\frac{125}{1440}$	$-\frac{68}{1440}$	$\frac{13}{1440}$	$-\frac{80}{1440}$	$\frac{64}{1440}$	$\frac{16}{1440}$

r k	1 1			1 -1		
ν \ μ	0	1	2	0	1	2
2	$\frac{37}{1440}$	$-\frac{332}{1440}$	$-\frac{125}{1440}$	$\frac{13}{1440}$	$\frac{52}{1440}$	$-\frac{5}{1440}$
1	$\frac{28}{1440}$	$-\frac{320}{1440}$	$-\frac{68}{1440}$	$-\frac{68}{1440}$	$-\frac{320}{1440}$	$\frac{28}{1440}$
0	$-\frac{5}{1440}$	$\frac{52}{1440}$	$\frac{13}{1440}$	$-\frac{125}{1440}$	$-\frac{332}{1440}$	$\frac{37}{1440}$

r k	2 -1			2 -2		
ν \ μ	0	1	2	0	1	2
2	$\frac{13}{1440}$	$-\frac{68}{1440}$	$-\frac{125}{1440}$	$\frac{16}{1440}$	$\frac{64}{1440}$	$-\frac{80}{1440}$
1	$\frac{52}{1440}$	$-\frac{320}{1440}$	$-\frac{332}{1440}$	$\frac{64}{1440}$	$-\frac{1280}{1440}$	$-\frac{704}{1440}$
0	$-\frac{5}{1440}$	$\frac{28}{1440}$	$\frac{37}{1440}$	$-\frac{80}{1440}$	$-\frac{704}{1440}$	$-\frac{176}{1440}$

Die Formeln (19) gelten gleichermaßen für Werte ober- und unterhalb $y=x$. Für $\lambda \geqq 1$ vereinfachen sich die Vorschriften (19) zu

$$
\begin{aligned}
z_{r,r+k}^{[\lambda+1]} &= z_{r,r+k}^{[\lambda]} + h^2 \sum_{\substack{\mu=0\\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} \eta_{\mu,\nu,m}^{r,k} \left(f_{\mu,\nu}^{[\lambda]} - f_{\mu,\nu}^{[\lambda-1]}\right),\\
p_{r,r+k}^{[\lambda+1]} &= p_{r,r+k}^{[\lambda]} + h \sum_{\substack{\nu=0\\ \nu\neq r}}^{m} \eta_{\nu,m}^{r,k} \left(f_{r,\nu}^{[\lambda]} - f_{r,\nu}^{[\lambda-1]}\right),\\
q_{r,r+k}^{[\lambda+1]} &= q_{r,r+k}^{[\lambda]} - h \sum_{\substack{\mu=0\\ \mu\neq r+k}}^{m} \eta_{\mu,m}^{r,k} \left(f_{\mu,r+k}^{[\lambda]} - f_{\mu,r+k}^{[\lambda-1]}\right),\\
f_{r,r+k}^{[\lambda+1]} &= f\left(x_r, y_{r+k}, z_{r,r+k}^{[\lambda+1]}, p_{r,r+k}^{[\lambda+1]}, q_{r,r+k}^{[\lambda+1]}\right).
\end{aligned}
\tag{21}
$$

Für die exakte Lösung der Differentialgleichung gilt:

$$
\begin{aligned}
z(x_r, y_{r+k}) &= z(x_r, y_r) + h \sum_{\mu=0}^{m} \eta_{\mu,m}^{r,k} q(x_\mu, y_\mu) + \\
&\quad + h^2 \sum_{\mu=0}^{m} \sum_{\nu=0}^{m} \eta_{\mu,\nu,m}^{r,k} F(x_\mu, y_\nu) + S_{m+1,m+1}^{r,r+k} + \overset{\equiv}{S}{}_{m+1}^{r,r+k},\\
p(x_r, y_{r+k}) &= p(x_r, y_r) + h \sum_{\nu=0}^{m} \eta_{\nu,m}^{r,k} F(x_r, y_\nu) + \bar{S}_{m+1}^{r,r+k},\\
q(x_r, y_{r+k}) &= q(x_{r+k}, y_{r+k}) - h \sum_{\mu=0}^{m} \eta_{\mu,m}^{r,k} F(x_\mu, y_{r+k}) + \bar{\bar{S}}{}_{m+1}^{r,r+k}.
\end{aligned}
\tag{22}
$$

Für die Kubatur- und Quadraturfehler seien hier nur einfache aber gröbere Abschätzungen gegeben:

$$
\begin{aligned}
\left|S_{m+1,m+1}^{r,r+k}\right| &< \frac{h^{m+3}}{m+1}\,\frac{k^2}{2}\left[\left|\frac{\partial^{m+1}}{\partial x^{m+1}} F\right|_{\mathrm{Max}} + \left|\frac{\partial^{m+1}}{\partial y^{m+1}} F\right|_{\mathrm{Max}} + \right.\\
&\qquad \left. + \frac{h^{m+1}}{m+1}\left|\frac{\partial^{2m+2}}{\partial x^{m+1}\,\partial y^{m+1}} F\right|_{\mathrm{Max}}\right],\\
\left|\overset{\equiv}{S}{}_{m+1}^{r,r+k}\right| &< |k|\,\frac{h^{m+2}}{m+1}\left|\frac{\partial^{m}}{\partial x^{m}} F\right|_{\mathrm{Max}},\\
\left|\bar{S}_{m+1}^{r,r+k}\right| &< |k|\,\frac{h^{m+2}}{m+1}\left|\frac{\partial^{m+1}}{\partial y^{m+1}} F\right|_{\mathrm{Max}},\\
\left|\bar{\bar{S}}{}_{m+1}^{r,r+k}\right| &< |k|\,\frac{h^{m+2}}{m+1}\left|\frac{\partial^{m+1}}{\partial x^{m+1}} F\right|_{\mathrm{Max}}.
\end{aligned}
\tag{23}
$$

Bei der Berechnung der Gewichte (20) ergeben sich Vereinfachungen: Wie man leicht bestätigt, ist [siehe Teil I dieser Arbeit S. 434]:

$$\eta_{\mu,m}^{r,k} = \gamma_{r+k,\mu}^{m} - \gamma_{r,\mu}^{m}, \tag{24}$$

und für $t \geqq 1$

$$\eta_{\mu,m}^{r,-t} = -\,\eta_{\mu,m}^{r-t,t}. \tag{25}$$

Ferner gilt nach (20):

$$\eta_{\mu,\nu,m}^{r,k} = \eta_{\mu,\nu,m}^{0,r+k} - \eta_{\mu,\nu,m}^{0,r} + \eta_{\mu,m}^{0,r}\left(\eta_{\nu,m}^{0,r+k} - \eta_{\nu,m}^{0,r}\right). \tag{26}$$

Schließlich läßt sich die Berechnung der $\eta^{0,\alpha}_{\mu,\nu,m}$ noch vereinfachen: Ersetzt man in (20) nacheinander v durch $-v+\alpha$, u durch $-u+\alpha$, so wird

$$\eta^{0,\alpha}_{\mu,\nu,m} = \frac{(-1)^{\mu+\nu+1}}{\mu!\,\nu!\,(m-\mu)!\,(m-\nu)!} \int\limits_0^\alpha (u-\alpha)(u-\alpha+1)\dots(u-\alpha+\mu-1)\times$$
$$\times(u-\alpha+\mu+1)\dots(u-\alpha+m)\times \tag{27}$$
$$\times\left[\int\limits_0^u (v-\alpha)(v-\alpha+1)\dots(v-\alpha+\nu-1)(v-\alpha+\nu+1)\dots(v-\alpha+m)\,dv\right]du.$$

Mit Hilfe der Formeln (17), (18), (19), bzw. (21) errechnet man die Werte über dem in Abb. 1 (für $m=2$) mit „1" bezeichneten Quadrat. Die gleichen Formeln dienen aber auch zur Berechnung der Näherungswerte über den in Abb. 1 mit „2", „3", ... bezeichneten Quadraten. Man braucht mit der Zählung der Werte über den einzelnen Quadraten lediglich wieder von vorn anzufangen. So wird man z.B. $z_{r,r+k}$ anstelle von $z_{\alpha m+r,\alpha m+r+k}$ setzen, $(\alpha=1, 2, \dots)$.

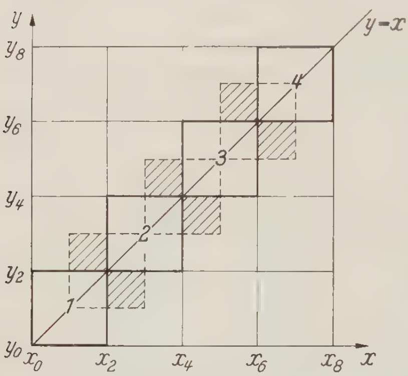

Abb. 1. Zur Berechnung des Anfangsfeldes

Die Werte über den Quadraten stellen dann ein Anfangsfeld dar. Alle anderen Werte kann man schon mit dem Verfahren für die fortlaufende Rechnung ermitteln, allerdings unter Benutzung von höchstens 10 Funktionswerten. Reicht die Genauigkeit nicht aus, so kann man sich mit der Berechnung weiterer Werte über „Zwischenquadraten" (in Abb. 1 schraffiert gezeichnet) helfen. Auch dazu können offenbar die hier angegebenen Formeln dienen.

5. Interpolationsformeln für die fortlaufende Rechnung

Es werden wieder gesondert Formeln zur Berechnung von Näherungswerten ober- und unterhalb $y=x$ entwickelt. Da die Betrachtungen hierfür denen in 3. sehr ähnlich verlaufen, können wir uns kurz fassen.

Mit $\xi=x_r$, $\eta=y_{s+1}$ wird in (4) ersetzt:

$$\begin{aligned}
&f(x,y,z,p,q) &&\text{durch} \quad \sum_{\mu=0}^{m}\sum_{\nu=0}^{m-\mu}\frac{U^{(\mu)}}{\mu!}\,\frac{(V-1)^{(-\nu)}}{\nu!}\,\Delta_x^\mu\nabla_y^\nu f_{r,s+1},\\
&q\big(x,x+(y_s-x_r)\big) &&\text{durch} \quad \sum_{\mu=0}^{m}\frac{U^{(\mu)}}{\mu!}\,\Delta_x^\mu q_r,\\
&f(x_r,y,z,p,q) &&\text{durch} \quad \sum_{\nu=0}^{m}\frac{(V-1)^{(-\nu)}}{\nu!}\,\nabla_y^\nu f_{r,s+1},\\
&f(x,y_{s+1},z,p,q) &&\text{durch} \quad \sum_{\mu=0}^{m}\frac{U^{(\mu)}}{\mu!}\,\Delta_x^\mu f_{r,s+1}.
\end{aligned} \tag{28}$$

Dabei ist

$$\begin{aligned}
U^{(\mu)} &= u(u-1)\dots(u-\mu+1),\\
(V-1)^{(-\nu)} &= (v-1)\,v\,(v+1)\dots(v+\nu-2).
\end{aligned} \tag{29}$$

Führt man dann in (4) die Integrationen aus und ersetzt die Differenzen durch Funktionswerte, so erhält man als Iterationsvorschriften die Interpolationsformeln:

$$
\begin{aligned}
z_{r,s+1}^{[\lambda+1]} &= z_{r,s} + h \sum_{\mu=0}^{m} \alpha_{m,\mu}^{*}\, q_{r+\mu,s+\mu} - h^2 \left[\alpha_{0,0}^{*\,m} f_{r,s+1}^{[\lambda]} + \sum_{\substack{\mu=0 \\ \mu,\nu \neq 0,0}}^{m} \sum_{\nu=0}^{m-\mu} \alpha_{\mu,\nu}^{*\,m} f_{r+\mu,s+1-\nu} \right], \\
p_{r,s+1}^{[\lambda+1]} &= p_{r,s} + h \left[\alpha_{m,0}^{*} f_{r,s+1}^{[\lambda]} + \sum_{\nu=1}^{m} \alpha_{m,\nu}^{*} f_{r,s+1-\nu} \right], \\
q_{r,s+1}^{[\lambda+1]} &= q_{r+1,s+1} - h \left[\alpha_{m,0}^{*} f_{r,s+1}^{[\lambda]} + \sum_{\mu=1}^{m} \alpha_{m,\mu}^{*} f_{r+\mu,s+1} \right], \\
f_{r,s+1}^{[\lambda+1]} &= f\left(x_r, y_{s+1}, z_{r,s+1}^{[\lambda+1]}, p_{r,s+1}^{[\lambda+1]}, q_{r,s+1}^{[\lambda+1]}\right),
\end{aligned}
\tag{30}
$$

mit

$$
\begin{aligned}
\alpha_{\mu,\nu}^{*\,m} &= (-1)^{\mu+\nu} \sum_{\varkappa=\mu}^{m-\nu} \sum_{\lambda=\nu}^{m-\varkappa} (-1)^{\varkappa} \beta_{\varkappa,\lambda}^{*} \binom{\varkappa}{\mu} \binom{\lambda}{\nu}, \\
\beta_{\varkappa,\lambda}^{*} &= \frac{(-1)^{\varkappa+\lambda}}{\varkappa!\,\lambda!} \int_0^1 (u-1)\,u\,(u+1)\dots(u+\mu-2) \times \\
&\quad \times \left[\int_0^u v\,(v-1)\dots(v-\nu+1)\,dv \right] du,
\end{aligned}
\tag{31}
$$

$\alpha_{\varkappa,\lambda}^{*}$ wie (15).

Tabelle 4. Gewichte $\beta_{\varkappa,\lambda}^{*}$:

$\beta_{0,0}^{*} = \frac{1}{2}$	$\beta_{1,0}^{*} = \frac{1}{6}$	$\beta_{2,0}^{*} = -\frac{1}{12}$
$\beta_{0,1}^{*} = -\frac{1}{6}$	$\beta_{1,1}^{*} = -\frac{1}{24}$	
$\beta_{0,2}^{*} = -\frac{1}{12}$		

Tabelle 5. Gewichte $\alpha_{\mu,\nu}^{*\,m}$:

$m=0$:	$\alpha_{0,0}^{0} = \frac{1}{2}$		
$m=1$:	$\alpha_{0,0}^{*\,1} = \frac{1}{6}$	$\alpha_{1,0}^{*\,1} = \frac{1}{6}$	
	$\alpha_{0,1}^{*\,1} = \frac{1}{6}$		
$m=2$:	$\alpha_{0,0}^{*\,2} = \frac{1}{24}$	$\alpha_{1,0}^{*\,2} = \frac{7}{24}$	$\alpha_{2,0}^{*\,2} = -\frac{2}{24}$
	$\alpha_{0,1}^{*\,2} = \frac{7}{24}$	$\alpha_{1,1}^{*\,2} = \frac{1}{24}$	
	$\alpha_{0,2}^{*\,2} = -\frac{2}{24}$		

Für die Berechnung der Näherungen unterhalb $y = x$ erhält man entsprechend die Gleichungen:

$$
\begin{aligned}
z_{r+1,s}^{[\lambda+1]} &= z_{r+1,s+1} - h \sum_{\mu=0}^{m} \alpha_{m,\mu}^{*}\, q_{r+1-\mu,s+1-\mu} - h^2 \left[\alpha_{0,0}^{*\,m} f_{r+1,s}^{[\lambda]} + \sum_{\substack{\mu=0 \\ \mu,\nu \neq 0,0}}^{m} \sum_{\nu=0}^{m-\mu} \alpha_{\mu,\nu}^{*\,m} f_{r+1-\mu,s+\nu} \right], \\
p_{r+1,s}^{[\lambda+1]} &= p_{r+1,s+1} - h \left[\alpha_{m,0}^{*} f_{r+1,s}^{[\lambda]} + \sum_{\nu=1}^{m} \alpha_{m,\nu}^{*} f_{r+1,s+\nu} \right], \\
q_{r+1,s}^{[\lambda+1]} &= q_{r,s} + h \left[\alpha_{m,0}^{*} f_{r+1,s}^{[\lambda]} + \sum_{\mu=1}^{m} \alpha_{m,\mu}^{*} f_{r+1-\mu,s} \right], \\
f_{r+1,s}^{[\lambda+1]} &= f\left(x_{r+1}, y_s, z_{r+1,s}^{[\lambda+1]}, p_{r+1,s}^{[\lambda+1]}, q_{r+1,s}^{[\lambda+1]}\right).
\end{aligned}
\tag{32}
$$

Für $\lambda \geqq 1$ vereinfachen sich die Iterationsvorschriften beträchtlich:

Aus (30) wird:

$$\left.\begin{aligned}
z^{[\lambda+1]}_{r,s+1} &= z^{[\lambda]}_{r,s+1} - h^2 \alpha^{*\,m}_{0,0} \left(f^{[\lambda]}_{r,s+1} - f^{[\lambda-1]}_{r,s+1}\right),\\
p^{[\lambda+1]}_{r,s+1} &= p^{[\lambda]}_{r,s+1} + h \alpha^{*}_{m,0} \left(f^{[\lambda]}_{r,s+1} - f^{[\lambda-1]}_{r,s+1}\right),\\
q^{[\lambda+1]}_{r,s+1} &= q^{[\lambda]}_{r,s+1} - h \alpha^{*}_{m,0} \left(f^{[\lambda]}_{r,s+1} - f^{[\lambda-1]}_{r,s+1}\right),\\
f^{[\lambda+1]}_{r,s+1} &= f\left(x_r, y_{s+1}, z^{[\lambda+1]}_{r,s+1}, p^{[\lambda+1]}_{r,s+1}, q^{[\lambda+1]}_{r,s+1}\right).
\end{aligned}\right\} \tag{33}$$

Aus (32) wird:

$$\left.\begin{aligned}
z^{[\lambda+1]}_{r+1,s} &= z^{[\lambda]}_{r+1,s} - h^2 \alpha^{*\,m}_{0,0} \left(f^{[\lambda]}_{r+1,s} - f^{[\lambda-1]}_{r+1,s}\right),\\
p^{[\lambda+1]}_{r+1,s} &= p^{[\lambda]}_{r+1,s} - h \alpha^{*}_{m,0} \left(f^{[\lambda]}_{r+1,s} - f^{[\lambda-1]}_{r+1,s}\right),\\
q^{[\lambda+1]}_{r+1,s} &= q^{[\lambda]}_{r+1,s} + h \alpha^{*}_{m,0} \left(f^{[\lambda]}_{r+1,s} - f^{[\lambda-1]}_{r+1,s}\right),\\
f^{[\lambda+1]}_{r+1,s} &= f\left(x_{r+1}, y_{s}, z^{[\lambda+1]}_{r+1,s}, p^{[\lambda+1]}_{r+1,s}, q^{[\lambda+1]}_{r+1,s}\right).
\end{aligned}\right\} \tag{34}$$

Für die exakte Lösung der Differentialgleichung gelten ebenfalls die Gleichungen (30) und (32), wenn dort anstelle der Näherungswerte die exakten Werte eingesetzt werden, mit den Kubatur- und Quadraturfehlern (s. (22)) ${}^1S^*_{m+1,m+1} + {}^1\overset{\equiv}{S}{}^*_{m+1}$, ${}^1\bar{S}^*_{m+1}$, ${}^1\bar{\bar{S}}{}^*_{m+1}$ für Werte oberhalb $y=x$ und ${}^2S^*_{m+1,m+1} + {}^2\overset{\equiv}{S}{}^*_{m+1}$, ${}^2\bar{S}^*_{m+1}$, ${}^2\bar{\bar{S}}{}^*_{m+1}$ für Werte unterhalb $y=x$. Es gelten die Abschätzungen:

$$\left.\begin{aligned}
&\left.\begin{matrix} \left|{}_1S^*_{m+1,m+1}\right| \\ \left|{}^2S^*_{m+1,m+1}\right| \end{matrix}\right\} \leqq h^{m+3} \sum_{\lambda=0}^{m+1} \left|\frac{\partial^{m+1}}{\partial x^{m+1-\lambda}\,\partial y^{\lambda}} F\right|_{\mathrm{Max}} \left|\beta^*_{m+1-\lambda,\lambda}\right|,\\
&\left.\begin{matrix} \left|{}^1\overset{\equiv}{S}{}^*_{m+1}\right| \\ \left|{}^2\overset{\equiv}{S}{}^*_{m+1}\right| \end{matrix}\right\} \leqq h^{m+2} \left|\frac{\partial^{m}}{\partial x^{m}} F\right|_{\mathrm{Max}} \cdot \left|\beta^*_{m+1}\right|,\\
&\left.\begin{matrix} \left|{}^1\bar{S}^*_{m+1}\right| \\ \left|{}^2\bar{S}^*_{m+1}\right| \end{matrix}\right\} \leqq h^{m+2} \left|\frac{\partial^{m+1}}{\partial y^{m+1}} F\right|_{\mathrm{Max}} \left|\beta^*_{m+1}\right|,\\
&\left.\begin{matrix} \left|{}^1\bar{\bar{S}}{}^*_{m+1}\right| \\ \left|{}^2\bar{\bar{S}}{}^*_{m+1}\right| \end{matrix}\right\} \leqq h^{m+2} \left|\frac{\partial^{m+1}}{\partial x^{m+1}} F\right|_{\mathrm{Max}} \left|\beta^*_{m+1}\right|.
\end{aligned}\right\} \tag{35}$$

Die Fehler der Näherungen für z, die nur durch genäherte Kubatur und Quadratur entstehen, haben die Form

$$\left.\begin{matrix} \left|{}^1S^*_{m+1,m+1}\right| + \left|{}^1\overset{\equiv}{S}{}^*_{m+1}\right| \\ \left|{}^2S^*_{m+1,m+1}\right| + \left|{}^2\overset{\equiv}{S}{}^*_{m+1}\right| \end{matrix}\right\} \leqq A\,h^{m+2} + B\,h^{m+3} = C\,h^{m+2}.$$

Formal erreicht man Proportionalität der Fehler zu h^{m+3}, wenn

in (30) $\quad \sum_{\mu=0}^{m+1} \alpha^*_{m+1,\mu}\, q_{r+\mu,s+\mu} \quad$ statt $\quad \sum_{\mu=0}^{m} \alpha^*_{m,\mu}\, q_{r+\mu,s+\mu}$,

in (32) $\quad \sum_{\mu=0}^{m+1} \alpha^*_{m+1,\mu}\, q_{r+1-\mu,s+1-\mu} \quad$ statt $\quad \sum_{\mu=0}^{m} \alpha^*_{m,\mu}\, q_{r+1-\mu,s+1-\mu}$ gesetzt wird.

Es können noch andere Interpolationsverfahren gefunden werden, je nach der Art der Approximation der Integranden in (4) und (5). Prinzipiell ergeben sich aber wohl keine neuen Gesichtspunkte.

Es sei schließlich noch erwähnt:

Kennt man bei der Behandlung des Cauchy-Problems schon eine Anzahl von Näherungswerten an bestimmten Stellen, so kann mit den Verfahren zur Lösung des charakteristischen Anfangswertproblems weitergerechnet werden (siehe Teil I).

6. Konvergenz der Anfangsiteration

Für die folgenden Betrachtungen genüge das Problem (1) den Voraussetzungen des Satz 1. Setzt man

$$\begin{gathered} z_{\varrho,\sigma}^{[\lambda+1]} - z_{\varrho,\sigma}^{[\lambda]} = \delta_{\varrho,\sigma}^{[\lambda]}, \qquad p_{\varrho,\sigma}^{[\lambda+1]} - p_{\varrho,\sigma}^{[\lambda]} = \bar{\delta}_{\varrho,\sigma}^{[\lambda]}, \qquad q_{\varrho,\sigma}^{[\lambda+1]} - q_{\varrho,\sigma}^{[\lambda]} = \bar{\bar{\delta}}_{\varrho,\sigma}^{[\lambda]}, \\ |\delta_{\varrho,\sigma}^{[\lambda]}| + |\bar{\delta}_{\varrho,\sigma}^{[\lambda]}| + |\bar{\bar{\delta}}_{\varrho,\sigma}^{[\lambda]}| = w_{\varrho,\sigma}^{[\lambda]}, \end{gathered} \tag{36}$$

so gelten für die Gleichungen (21) die Abschätzungen

$$\begin{gathered} w_{r,r+k}^{[\lambda]} \leqq h^2 M \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |\eta_{\mu,\nu,m}^{r,k}|\, w_{\mu,\nu}^{[\lambda-1]} + h M \left[\sum_{\substack{\nu=0 \\ \nu\neq r}}^{m} |\eta_{\nu,m}^{r,k}|\, w_{r,\nu}^{[\lambda-1]} + \sum_{\substack{\mu=0 \\ \mu\neq r+k}}^{m} |\eta_{\mu,m}^{r,k}|\, w_{\mu,r+k}^{[\lambda-1]} \right], \\ (r = 0, 1, \ldots, m;\; k = -r, -r+1, \ldots, -1, +1, \ldots, m-r). \end{gathered} \tag{37}$$

Dieses Ungleichungssystem hat die Form

$$w_{r,r+k}^{[\lambda]} \leqq \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |B_{\mu,\nu}^{r,k}|\, w_{\mu,\nu}^{[\lambda-1]}. \tag{38}$$

Sind alle Eigenwerte der Matrix $\mathfrak{B} = \left(|B_{\mu,\nu}^{r,k}|\right)$

dem Betrag nach < 1, so ist die Konvergenz der Reihen

$$\sum_{\lambda=0}^{\infty} \delta_{r,r+k}^{[\lambda]}, \qquad \sum_{\lambda=0}^{\infty} \bar{\delta}_{r,r+k}^{[\lambda]}, \qquad \sum_{\lambda=0}^{\infty} \bar{\bar{\delta}}_{r,r+k}^{[\lambda]} \tag{39}$$

gesichert. Hinreichende Bedingungen für die Konvergenz sind

$$\begin{aligned} &\operatorname*{Max}_{r,k} \left\{ \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |B_{\mu,\nu}^{r,k}| \right\} < 1, \qquad \text{(Zeilensummenkriterium)}, \\ &\operatorname*{Max}_{\mu,\nu} \left\{ \sum_{\substack{r=0 \\ k\neq 0}}^{m} \sum_{k=-r}^{m-r} |B_{\mu,\nu}^{r,k}| \right\} < 1, \qquad \text{(Spaltensummenkriterium)}. \end{aligned} \tag{40}$$

Sei

$$\operatorname*{Max}_{r,k} \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |\eta_{\mu,\nu,m}^{r,k}| = A, \qquad \operatorname*{Max}_{r,k} \sum_{\substack{\nu=0 \\ \nu\neq r}}^{m} |\eta_{\nu,m}^{r,k}| = B, \qquad \operatorname*{Max}_{r,k} \sum_{\substack{\mu=0 \\ \mu\neq r+k}}^{m} |\eta_{\mu,m}^{r,k}| = C,$$

so ist nach (37), (38) und (40) sicher

$$h^2 M A + h M (B + C) < 1 \tag{41}$$

hinreichend für die Konvergenz. Sei ferner

$$\operatorname{Max}\{A, B, C\} = L,$$

dann ist

$$h < \sqrt{1 + \frac{1}{ML}} - 1 \tag{42}$$

ein weiteres hinreichendes Konvergenzkriterium.

Für $m=2$ lauten die Bedingungen:

Nach (40):

$$h < \frac{25}{18}\left(\sqrt{1+\frac{54}{125M}}-1\right),$$

$$h < \frac{75}{52}\left(\sqrt{1+\frac{208}{375M}}-1\right).$$

Nach (41):
$$h < \frac{25}{18}\left(\sqrt{1+\frac{54}{125M}}-1\right). \tag{43}$$

Nach (42):

$$h < \sqrt{1+\frac{3}{5M}}-1.$$

7. Konvergenz der Iterationen bei der fortlaufenden Rechnung

Mit den Bezeichnungen (36) gelten für die Gleichungen (33) die Abschätzungen:

$$w_{r,s+1}^{[\lambda]} \leqq \left(h^2 M\,|\alpha_{0,0}^{*\,m}| + 2hM\,|\alpha_{m,0}^{*}|\right) w_{r,s+1}^{[\lambda-1]}. \tag{44}$$

Ein hinreichendes Konvergenzkriterium für das Verfahren (33) bzw. (30) und — wie man sofort übersieht — auch für (32) bzw. (34) ist daher:

$$M\left(h^2\,|\alpha_{0,0}^{*\,m}|\,2h + |\alpha_{m,0}^{*}|\right) < 1. \tag{45}$$

Hieraus erhält man noch

$$h < \frac{|\alpha_{m,0}^{*}|}{|\alpha_{0,0}^{*\,m}|}\left(\sqrt{1+\frac{|\alpha_{0,0}^{*\,m}|}{M(\alpha_{m,0}^{*})^2}}-1\right). \tag{46}$$

Für $m=2$ ist

$$h < 10\left(\sqrt{1+\frac{6}{25M}}-1\right). \tag{47}$$

8. Fehlerabschätzung für die Näherungswerte des Anfangsfeldes

Es werde vorausgesetzt, daß die Iterationen (21) jeweils bis zum Stillstand durchgeführt wurden, so daß die Indizes $[\lambda]$ fortgelassen werden können. Abrundungsfehler werden nicht berücksichtigt.

Der Einfachheit halber sei der Quadraturfehler $\bar{\bar{S}}_{m+1}^{r,r+k}$ in (22) zum Kubaturfehler $S_{m+1,m+1}^{r,r+k}$ hinzugenommen, so daß im folgenden statt $S_{m+1,m+1}^{r,r+k} + \bar{\bar{S}}_{m+1}^{r,r+k}$ nur $S_{m+1,m+1}^{r,r+k}$ geschrieben wird. Subtrahiert man die Gleichungen (22) von den Gleichungen (19), so gelten mit den Bezeichnungen

$$z_{\mu,\nu} - z(x_\mu, y_\nu) = \varepsilon_{\mu,\nu}, \qquad p_{\mu,\nu} - p(x_\mu, y_\nu) = \bar{\varepsilon}_{\mu,\nu}, \qquad q_{\mu,\nu} - q(x_\mu, y_\nu) = \bar{\bar{\varepsilon}}_{\mu,\nu} \tag{48}$$

die Abschätzungen:

$$\begin{aligned}
&|\varepsilon_{r,r+k}| - h^2 M \sum_{\substack{\mu=0\\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |\eta_{\mu,\nu,m}^{r,k}| \left\{|\varepsilon_{\mu,\nu}| + |\bar{\varepsilon}_{\mu,\nu}| + |\bar{\bar{\varepsilon}}_{\mu,\nu}|\right\} \leqq |S_{m+1,m+1}^{r,r+k}|,\\
&|\bar{\varepsilon}_{r,r+k}| - hM \sum_{\substack{\nu=0\\ \nu\neq r}}^{m} |\eta_{\nu,m}^{r,k}| \left\{|\varepsilon_{r,\nu}| + |\bar{\varepsilon}_{r,\nu}| + |\bar{\bar{\varepsilon}}_{r,\nu}|\right\} \leqq |\bar{S}_{m+1}^{r,r+k}|,\\
&|\bar{\bar{\varepsilon}}_{r,r+k}| - hM \sum_{\substack{\mu=0\\ \mu\neq r+k}}^{m} |\eta_{\mu,m}^{r,k}| \left\{|\varepsilon_{\mu,r+k}| + |\bar{\varepsilon}_{\mu,r+k}| + |\bar{\bar{\varepsilon}}_{\mu,r+k}|\right\} \leqq |\bar{\bar{S}}_{m+1}^{r,r+k}|,
\end{aligned} \tag{49}$$

mit den Abschätzungen (23) für die Kubatur- und Quadraturfehler. Die Matrix des Systems (49) ist von monotoner Art [5], mit

$$\begin{array}{lll} |\varepsilon_{\varkappa,\lambda}| \leqq |Z_{\varkappa,\lambda}, & |\bar{\varepsilon}_{\varkappa,\lambda}| \leqq \bar{Z}_{\varkappa,\lambda}, & |\bar{\bar{\varepsilon}}_{\varkappa,\lambda}| \leqq |\bar{\bar{Z}}_{\varkappa,\lambda}, \\ |S^{r,r+k}_{m+1,m+1}| \leqq T^{r,r+k}_{m+1,m+1}, & |\bar{S}^{r,r+k}_{m+1}| \leqq \bar{T}^{r,r+k}_{m+1}, & |\bar{\bar{S}}^{r,r+k}_{m+1}| \leqq \bar{\bar{T}}^{r,r+k}_{m+1}, \end{array} \tag{50}$$

gilt daher:

$$\begin{aligned} &Z_{r,r+k} - h^2 M \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} |\eta^{r,k}_{\mu,\nu,m}| \{Z_{\mu,\nu} + \bar{Z}_{\mu,\nu} + \bar{\bar{Z}}_{\mu,\nu}\} = T^{r,r+k}_{m+1,m+1}, \\ &\bar{Z}_{r,r+k} - h M \sum_{\substack{\nu=0 \\ \nu\neq r}}^{m} |\eta^{r,k}_{\nu,m}| \{Z_{r,\nu} + \bar{Z}_{r,\nu} + \bar{\bar{Z}}_{r,\nu}\} = \bar{T}^{r,r+k}_{m+1}, \\ &\bar{\bar{Z}}_{r,r+k} - h M \sum_{\substack{\mu=0 \\ \mu\neq r+k}}^{m} |\eta^{r,k}_{\mu,m}| \{Z_{\mu,r+k} + \bar{Z}_{\mu,r+k} + \bar{\bar{Z}}_{\mu,r+k}\} = \bar{\bar{T}}^{r,r+k}_{m+1}. \end{aligned} \tag{51}$$

Das Gleichungssystem (51) hat die Form

$$\mathfrak{z} - \mathfrak{B}\,\mathfrak{z} = \mathfrak{T}$$

und mit der Einheitsmatrix $\mathfrak{E}$ die nach (40) konvergente Matrizenreihe

$$\mathfrak{z} = (\mathfrak{E} + \mathfrak{B} + \mathfrak{B}^2 + \cdots)\,\mathfrak{T}$$

als Lösung. Sei

$$\operatorname{Max}\{|\eta^{r,k}_{\nu,m}|\} = a, \qquad \operatorname{Max}_{r,k}\left\{\sum_{\substack{\lambda=0 \\ \lambda\neq r}}^{m} |\eta^{r,k}_{\lambda,m}|\right\} = A,$$

so gilt bei genügend kleinem h für das größte Element $\operatorname{Max}(\mathfrak{B})$ der Matrix $\mathfrak{B}$:

$$\operatorname{Max}(\mathfrak{B}) = h M a$$

und allgemein für das größte Element $\operatorname{Max}(\mathfrak{B}^n)$ der Matrix $\mathfrak{B}^n$:

$$\operatorname{Max}(\mathfrak{B}^n) \leqq h^n M^n a A^{n-1}.$$

Für die Näherungswerte des Anfangsfeldes gilt dann die Fehlerabschätzung:

$$\begin{aligned} |\varepsilon_{r,r+k}| &\leqq T^{r,r+k}_{m+1,m+1} + \frac{M h a}{1 - h M A} \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} \left(T^{\mu,\nu}_{m+1,m+1} + \bar{T}^{\mu,\nu}_{m+1} + \bar{\bar{T}}^{\mu,\nu}_{m+1}\right), \\ |\bar{\varepsilon}_{r,r+k}| &\leqq \bar{T}^{r,r+k}_{m+1} + \frac{h M a}{1 - h M A} \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} \left(T^{\mu,\nu}_{m+1,m+1} + \bar{T}^{\mu,\nu}_{m+1} + \bar{\bar{T}}^{\mu,\nu}_{m+1}\right), \\ |\bar{\bar{\varepsilon}}_{r,r+k}| &\leqq \bar{\bar{T}}^{r,r+k}_{m+1} + \frac{h M a}{1 - h M A} \sum_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} \sum_{\nu=0}^{m} \left(T^{\mu,\nu}_{m+1,m+1} + \bar{T}^{\mu,\nu}_{m+1} + \bar{\bar{T}}^{\mu,\nu}_{m+1}\right). \end{aligned} \tag{52}$$

Damit ist die Fehlerabschätzung im Prinzip geleistet.

Mit Hilfe von (52) läßt sich zeigen: Zu beliebig vorgegebenem $\varepsilon > 0$ läßt sich stets eine Schrittweite h so angeben, daß für $r = 0, 1, \ldots, m$; $k = -r, -r+1, \ldots -1, +1 \ldots, m-r$:

$$|\varepsilon_{r,r+k}| < \varepsilon, \qquad |\bar{\varepsilon}_{r,r+k}| < \varepsilon, \qquad |\bar{\bar{\varepsilon}}_{r,r+k}| < \varepsilon.$$

9. Fehlerbetrachtungen beim Interpolationsverfahren der fortlaufenden Rechnung

Es seien die Voraussetzungen aus 8. erfüllt und obere Schranken der Fehlerbeträge des Anfangsfeldes bekannt. Die folgenden Fehlerbetrachtungen werden für die Formeln (30) zur Berechnung von Näherungswerten oberhalb $y=x$ durchgeführt. Für unterhalb $y=x$ gilt Entsprechendes.

Mit den Bezeichnungen (48) gelten dann für die Fehler die Abschätzungen:

$$\begin{aligned}
&|\varepsilon_{r,s+1}| - h^2 M\,|\alpha^{*\,m}_{0,0}|\left\{|\varepsilon_{r,s+1}| + |\bar{\varepsilon}_{r,s+1}| + |\bar{\bar{\varepsilon}}_{r,s+1}|\right\} \leqq A_m,\\
&|\bar{\varepsilon}_{r,s+1}| - h M\,|\alpha^{*}_{m,0}|\left\{|\varepsilon_{r,s+1}| + |\bar{\varepsilon}_{r,s+1}| + |\bar{\bar{\varepsilon}}_{r,s+1}|\right\} \leqq \bar{A}_m,\\
&|\bar{\bar{\varepsilon}}_{r,s+1}| - h M\,|\alpha^{*}_{m,0}|\left\{|\varepsilon_{r,s+1}| + |\bar{\varepsilon}_{r,s+1}| + |\bar{\bar{\varepsilon}}_{r,s+1}|\right\} \leqq \bar{\bar{A}}_m,
\end{aligned}\tag{53}$$

mit

$$\begin{aligned}
A_m &= |\varepsilon_{r,s}| + h\sum_{\mu=0}^{m}|\alpha^{*}_{m,\mu}|\,|\bar{\bar{\varepsilon}}_{r+\mu,s+\mu}| + h^2 M \sum_{\substack{\mu=0\\ \mu,\nu \neq 0,0}}^{m}\ \sum_{\nu=0}^{m}{}^{\mu}\,|\alpha^{*\,m}_{\mu,\nu}| \times\\
&\quad\times\left\{|\varepsilon_{r+\mu,s+1-\nu}| + |\bar{\varepsilon}_{r+\mu,s+1-\nu}| + |\bar{\bar{\varepsilon}}_{r+\mu,s+1-\nu}|\right\} + |{}^1S^{*}_{m+1,m+1}| + |{}^1\bar{\bar{S}}^{*}_{m+1}|,\\
\bar{A}_m &= |\bar{\varepsilon}_{r,s}| + h M\sum_{\nu=1}^{m}|\alpha^{*}_{m,\nu}|\left\{|\varepsilon_{r,s+1-\nu}| + |\bar{\varepsilon}_{r,s+1-\nu}| + |\bar{\bar{\varepsilon}}_{r,s+1-\nu}|\right\} + |{}^1\bar{S}^{*}_{m+1}|,\\
\bar{\bar{A}}_m &= |\bar{\bar{\varepsilon}}_{r+1,s+1}| + h M\sum_{\mu=1}^{m}|\alpha^{*}_{m,\mu}|\left\{|\varepsilon_{r+\mu,s+1}| + |\bar{\varepsilon}_{r+\mu,s+1}| + |\bar{\bar{\varepsilon}}_{r+\mu,s+1}|\right\} + |{}^{1*}_{m+1}\bar{\bar{S}}|.
\end{aligned}\tag{54}$$

Die A_m, $\bar{A}_m$, $\bar{\bar{A}}_m$ sind nach Voraussetzung bekannt, ihre Berechnung allerdings ist mühsam. Es lassen sich aber obere Schranken angeben, die einfacher berechnet werden können:

Sei

$$\operatorname{Max}\left\{|\varepsilon_{r+\mu,s+1-\nu}|\right\} = Z,\quad \operatorname{Max}\left\{|\bar{\varepsilon}_{r+\mu,s+1-\nu}|\right\} = \bar{Z},\quad \operatorname{Max}\left\{|\bar{\bar{\varepsilon}}_{r+\mu,s+1-\nu}|\right\} = \bar{\bar{Z}},$$
$$(\mu = 0, 1, \ldots, m;\ \nu = 0, 1, \ldots, m-\mu;\ \mu, \nu, \neq 0, 0),$$

ferner

$$\sum_{\substack{\mu=0\\ \mu,\nu\neq 0,0}}^{m}\ \sum_{\nu=0}^{m-\mu}|\alpha^{*\,m}_{\mu,\nu}| = G(m) = G,\qquad \sum_{\mu=1}^{m}|\alpha^{*}_{m,\mu}| = L(m) = L,$$

so gelten die Abschätzungen

$$\begin{aligned}
A_m &\leqq |\varepsilon_{r,s}| + h\left(L + |\alpha^{*}_{m,0}|\,\bar{\bar{Z}}\right) + h^2 M G\{Z + \bar{Z} + \bar{\bar{Z}}\} + |{}^1S^{*}_{m+1}| + |{}^1\bar{\bar{S}}^{*}_{m+1,m+1}| = R_m,\\
\bar{A}_m &\leqq |\bar{\varepsilon}_{r,s}| + h M L\{Z + \bar{Z} + \bar{\bar{Z}}\} + |\bar{S}1^{*}_{m+1}| = \bar{R}_m,\\
\bar{\bar{A}}_m &\leqq |\bar{\bar{\varepsilon}}_{r+1,s+1}| + h M L\{Z + \bar{Z} + \bar{\bar{Z}}\} + |{}^1\bar{\bar{S}}^{*}_{m+1}| = \bar{\bar{R}}_m\,.
\end{aligned}\tag{55}$$

Unter Berücksichtigung der Monotonie der Matrix des Systems (53) erhält man dann die Fehlerabschätzungen:

$$\begin{aligned}
&|\varepsilon_{r,s+1}| \leqq R_m + \frac{h^2 M\,|\alpha^{*\,m}_{0,0}|}{1 - h^2 M\,|\alpha^{*\,m}_{0,0}| - 2h M\,|\alpha^{*}_{m,0}|}\,(R_m + \bar{R}_m + \bar{\bar{R}}_m),\\
&\left.\begin{matrix}|\bar{\varepsilon}_{r,s+1}|\\ |\bar{\bar{\varepsilon}}_{r,s+1}|\end{matrix}\right\} \leqq \left\{\begin{matrix}\bar{R}_m\\ \bar{\bar{R}}_m\end{matrix}\right. + \frac{h M\,|\alpha^{*}_{m,0}|}{1 - h^2 M\,|\alpha^{*\,m}_{0,0}| - 2h M\,|\alpha^{*}_{m,0}|}\,(R_m + \bar{R}_m + \bar{\bar{R}}_m).
\end{aligned}\tag{56}$$

Damit ist gezeigt, daß wenigstens prinzipiell eine Fehlerabschätzung möglich ist, wenn obere Schranken der Fehlerbeträge vorher berechneter Näherungen bekannt sind.

10. Ein Konvergenzsatz

Es wird im folgenden gezeigt, daß bei der numerischen Behandlung des Cauchy-Problems die mit dem Interpolationsverfahren für die fortlaufende Rechnung gewonnenen Näherungswerte bei kleiner werdender Schrittweite gegen die Werte der exakten Lösung des Problems (3) konvergieren. Für die Extrapolationsverfahren läßt sich der Beweis ganz ähnlich führen.

Es sei das Anfangsfeld berechnet, (eventuell mit einem anderen Verfahren als dem in dieser Arbeit beschriebenem), d.h. es sind Näherungswerte

$$z_{\lambda+r,\lambda+r+k}, \quad p_{\lambda+r,\lambda+r+k}, \quad q_{\lambda+r,\lambda+r+k},$$
$$(\lambda=0, m, 2m, \ldots;\ r=0, 1, \ldots, m;\ k=-r, -r+1, \ldots, -1, +1, \ldots, m-r), \tag{57}$$

bekannt. Die zugehörigen Fehlerbeträge seien

$$|\varepsilon_{\lambda+r,\lambda+r+k}|, \quad |\bar{\varepsilon}_{\lambda+r,\lambda+r+k}|, \quad |\bar{\bar{\varepsilon}}_{\lambda+r,\lambda+r+k}|, \tag{58}$$

ferner

$$\operatorname{Max}\{|\varepsilon_{\lambda+r,\lambda+r+k}|\}=E_0, \quad \operatorname{Max}\{|\bar{\varepsilon}_{\lambda+r,\lambda+r+k}|\}=\bar{E}_0, \quad \operatorname{Max}\{|\bar{\bar{\varepsilon}}_{\lambda+r,\lambda+r+k}|\}=\bar{\bar{E}}_0. \tag{59}$$

Es werde nun vorausgesetzt:

$$\lim_{h\to 0} E_0 = \lim_{h\to 0} \bar{E}_0 = \lim_{h\to 0} \bar{\bar{E}}_0 = 0. \tag{60}$$

Schließlich seien die in den Abschätzungen für die Kubatur- und Quadraturfehler vorkommenden Ableitungen höherer Ordnung von $F(x, y)$ beschränkt. Dann gilt der

Satz 2. *Bei der numerischen Behandlung des Cauchy-Problems seien*

$$z_{r,s}, \ p_{r,s}, \ q_{r,s}, \qquad (r=0, 1, 2, \ldots;\ s=0, 1, 2, \ldots;\ r \neq s),$$

die mit dem Interpolationsverfahren der fortlaufenden Rechnung ermittelten Näherungswerte der exakten Lösungswerte

$$z(x_r, y_s), \quad p(x_r, y_s), \quad q(x_r, y_s)$$

des Problems (3). *Dann ist bei beliebigen aber festen* α, β, *wenn* $z(\alpha,\beta)$, $p(\alpha,\beta)$, $q(\alpha,\beta)$ *die exakten Werte und* $z_{\alpha,\beta}$, $p_{\alpha,\beta}$, $q_{\alpha,\beta}$ *die Näherungswerte sind:*

$$z_{\alpha,\beta}-z(\alpha,\beta)=\varepsilon_{\alpha,\beta}\to 0, \quad p_{\alpha,\beta}-p(\alpha,\beta)=\bar{\varepsilon}_{\alpha,\beta}\to 0, \quad q_{\alpha,\beta}-q(\alpha,\beta)=\bar{\bar{\varepsilon}}_{\alpha,\beta}\to 0$$
$$\text{für} \quad h\to 0.$$

Beweis. Es werden alle Näherungen der z, p, q, die längs der Geraden $y=x\pm(\mu+1)h$, $(\mu=1, 2, 3, \ldots)$, mit dem Interpolationsverfahren für die fortlaufende Rechnung ermittelt werden — die also nicht zum Anfangsfeld gehören — mit z_μ, p_μ, q_μ bezeichnet. Die zugehörigen Fehlerbeträge seien $|\varepsilon_\mu|$, $|\bar{\varepsilon}_\mu|$, $|\bar{\bar{\varepsilon}}_\mu|$.

Sei

$$\mathfrak{x}_0 = \begin{pmatrix} E_0 \\ \overline{E}_0 \\ \overline{\overline{E}}_0 \end{pmatrix}, \qquad \mathfrak{z}_\mu = \begin{pmatrix} |\varepsilon_\mu| \\ |\overline{\varepsilon}_\mu| \\ |\overline{\overline{\varepsilon}}_\mu| \end{pmatrix},$$

ferner (61)

$$\mathfrak{s}_\mu = h^{m+2} \begin{pmatrix} |S_\mu| \\ |\overline{S}_\mu| \\ |\overline{\overline{S}}_\mu| \end{pmatrix}$$

der Vektor, dessen Komponenten die maximalen Beträge aller Kubatur- und Quadraturfehler sind, die bei der Berechnung der z_μ, p_μ, q_μ längs der Geraden $y = x \pm (\mu + 1)\, h$ entstehen, so gilt, wie man leicht erkennt, für das Interpolationsverfahren der fortlaufenden Rechnung eine Abschätzung der Form:

$$\mathfrak{A}\,\mathfrak{z}_1 \leqq \mathfrak{B}\,\mathfrak{x}_0 + \mathfrak{s}_1 = \mathfrak{r}, \tag{62}$$

mit

$$\mathfrak{B} = \mathfrak{E} + h \begin{pmatrix} b_{11}(h) & b_{12}(h) & b_{13}(h) \\ b_{21}(h) & b_{22}(h) & b_{23}(h) \\ b_{31}(h) & b_{32}(h) & b_{33}(h) \end{pmatrix}, \quad 0 < b_{ik}(h) < B \quad \text{für} \quad h < H, \tag{63}$$

$$\mathfrak{A} = \mathfrak{E} - h \begin{pmatrix} h\,a_1 & h\,a_1 & h\,a_1 \\ a_2 & a_2 & a_2 \\ a_2 & a_2 & a_2 \end{pmatrix}, \quad 0 < a_1, \quad a_2 < A, \quad 0 < \operatorname{Det} \mathfrak{A}. \tag{64}$$

Die Matrix $\mathfrak{A}$ ist monoton, denn es existieren $\mathfrak{x}_1$ und der vom Nullvektor verschiedene Vektor $\mathfrak{r} > 0$, so daß

$$\mathfrak{A}\,\mathfrak{x}_1 = \mathfrak{r}. \tag{65}$$

Sei

$$\mathfrak{x}_1 = \begin{pmatrix} E_1 \\ \overline{E}_1 \\ \overline{\overline{E}}_1 \end{pmatrix},$$

so folgt wegen der Monotonie von $\mathfrak{A}$ aus (63) und (65) für $h \geqq 0$:

$$\mathfrak{z}_1 \leq \mathfrak{x}_1, \quad \text{also} \quad |\varepsilon_1| \leqq E_1, \quad |\overline{\varepsilon}_1| \leqq \overline{E}_1, \quad |\overline{\overline{\varepsilon}}_1| \leqq \overline{\overline{E}}_1.$$

Nach (65) ist

$$\mathfrak{A}\,\mathfrak{x}_1 = \mathfrak{A}\,\mathfrak{x}_0 + (\mathfrak{B} - \mathfrak{A})\,\mathfrak{x}_0 + \mathfrak{s}_1. \tag{66}$$

Da alle Elemente der Matrix $\mathfrak{B} - \mathfrak{A}$ für $h > 0$ positiv sind, so folgt:

$$\mathfrak{a}\,\varrho_0 \geqq \mathfrak{a}\,\varrho_1 \quad \text{und daher} \quad \varrho_0 \geqq \varrho_1$$

wegen der Monotonie von $\mathfrak{A}$.

Das soeben beschriebene Verfahren zur Ermittlung oberer Fehlerschranken kann fortgesetzt werden. Allgemein gilt für die Fehler der Näherungen längs der Geraden $y = x \pm (n+1)\, h$:

$$\mathfrak{A}\,\mathfrak{x}_n = \mathfrak{B}\,\mathfrak{x}_{n-1} + \mathfrak{s}_n, \qquad (n = 1, 2, \ldots), \tag{67}$$

mit

$$\mathfrak{x}_n = \begin{pmatrix} E_n \\ \overline{E}_n \\ \overline{\overline{E}}_n \end{pmatrix}, \quad \varrho_{n-1} \leqq \varrho_n, \quad |\varepsilon_n| \leqq E_n, \quad |\overline{\varepsilon}_n| \leqq \overline{E}_n, \quad |\overline{\overline{\varepsilon}}_n| \leqq \overline{\overline{E}}_n.$$

Wird $\mathfrak{A}^{-1}\mathfrak{B} = \mathfrak{C}$ gesetzt, so folgt aus (67), wie man durch vollständige Induktion bestätigt:

$$\mathfrak{x}_n = \mathfrak{C}^n \mathfrak{x}_0 + \sum_{\mu=0}^{n-1} \mathfrak{C}^\mu \mathfrak{A}^{-1} \mathfrak{s}_{n-\mu}, \quad \text{mit} \quad \mathfrak{C}^0 = \mathfrak{E}. \tag{68}$$

Die Vektoren $\mathfrak{A}^{-1}\mathfrak{s}_{n-\mu}$ müssen Darstellungen der Form

$$\mathfrak{A}^{-1} \mathfrak{s}_{n-\mu} = h^{m+2} \begin{pmatrix} \sigma_{n-\mu}(h) \\ \overline{\sigma}_{n-\mu}(h) \\ \overline{\overline{\sigma}}_{n-\mu}(h) \end{pmatrix}$$

besitzen mit

$$|\sigma_{n-\mu}(h)| < S_{n-\mu}, \quad |\overline{\sigma}_{n-\mu}(h)| < \overline{S}_{n-\mu}, \quad |\overline{\overline{\sigma}}_{n-\mu}(h)| < \overline{\overline{S}}_{n-\mu} \quad \text{für} \quad h < H.$$

Die Matrix $\mathfrak{A}$ hat nach (64) die Form

$$\mathfrak{A} = \mathfrak{E} - \overline{\mathfrak{A}}.$$

Die Matrizenreihe

$$\mathfrak{E} + \overline{\mathfrak{A}} + \overline{\mathfrak{A}}^2 + \overline{\mathfrak{A}}^3 + \cdots$$

konvergiert, wenn sämtliche Eigenwerte der Matrix $\overline{\mathfrak{A}}$ dem Betrag nach < 1 sind, was nach (45) stets erfüllt ist. Sie stellt dann die Matrix

$$\mathfrak{A}^{-1} = (\mathfrak{E} - \overline{\mathfrak{A}})^{-1}$$

dar. Die Matrix $\mathfrak{C}$ kann dargestellt werden durch

$$\mathfrak{C} = \mathfrak{E} + h \begin{pmatrix} c_{11}(h) & c_{12}(h) & c_{13}(h) \\ c_{21}(h) & c_{22}(h) & c_{23}(h) \\ c_{31}(h) & c_{32}(h) & c_{33}(h) \end{pmatrix}, \quad 0 < c_{ik} < C.$$

Bezeichnet man das größte Element der Matrix $\mathfrak{C}^\mu$ mit $\operatorname{Max}(\mathfrak{C}^\mu)$, so gilt:

$$\begin{aligned} &\operatorname{Max}(\mathfrak{C}) < 1 + hC \\ &\operatorname{Max}(\mathfrak{C}^2) < (1 + hC)(1 + 3hC), \\ &\vdots \\ &\operatorname{Max}(\mathfrak{C}^\mu) < (1 + hC)(1 + 3hC)^{\mu-1} < (1 + 3hC)^\mu. \end{aligned}$$

Betrachtet man jetzt die Fehlermajoranten auf den festen Geraden $y = x \pm k$ mit $(n+1)h = k$, so gilt mit $nh = k - h < K$ für alle $\mu \leqq n$:

$$\operatorname{Max}(\mathfrak{C}^\mu) \leqq (1 + 3hC)^{K/h} \leqq e^{3CK} \quad \text{für} \quad h \geqq 0. \tag{69}$$

Nach (68) folgt daher

$$\left.\begin{matrix} E_n \\ \overline{E}_n \\ \overline{\overline{E}}_n \end{matrix}\right\} < e^{3CK} [E_0 + \overline{E}_0 + \overline{\overline{E}}_0 + h^{m+1} K (S + \overline{S} + \overline{\overline{S}}) \tag{70}$$

mit

$$\mathrm{Max}\{S_{n-\mu}\} = S, \quad \mathrm{Max}\{\bar{S}_{n-\mu}\} = \bar{S}, \quad \mathrm{Max}\{\bar{\bar{S}}_{n-\mu}\} = \bar{\bar{S}}, \quad (\mu = 1, 2, \ldots, n-1).$$

Wegen der Beschränktheit der höheren Ableitungen von $F(x, y)$ sind auch die S, $\bar{S}$, $\bar{\bar{S}}$ beschränkt. Nach den Voraussetzungen am Anfang dieser Nummer folgt dann sogleich

$$\lim_{h\to 0} E_n = 0, \quad \lim_{h\to 0} \bar{E}_n = 0, \quad \lim_{h\to 0} \bar{\bar{E}}_n = 0,$$

und daher auch

$$\lim_{h\to 0} |\varepsilon_n| = 0, \quad \lim_{h\to 0} |\bar{\varepsilon}_n| - 0, \quad \lim_{h\to 0} |\bar{\bar{\varepsilon}}_n| - 0,$$

und daraus die Behauptung des Satzes.

Bei der Extrapolation ist $\mathfrak{A} = \mathfrak{A}^{-1} = \mathfrak{E}$ und somit $\mathfrak{C} = \mathfrak{B}$. Der Beweis läßt sich hier fast wörtlich genau so führen wie für die Interpolation.

11. Beispiel

Differentialgleichung:

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{1}{z} + z \quad \text{oder} \quad s = \frac{p\, q}{z} + z.$$

Anfangsbedingungen:

$$z(x, x) = e^{2x^2} \cos x,$$
$$p(x, x) = 3\, x\, e^{2x^2} \cos x,$$
$$q(x, x) = e^{2x^2} (x \cos x - \sin x).$$

Lösung:

$$z = e^{x^2 + xy} \cos y.$$

Ergebnis der Rechnung:

Das Anfangsfeld (umrandet) wurde mit $m=2$, $h=0,1$ nach (17), (18) und (19) bzw. (21) berechnet, die fortlaufende Rechnung mit Verfahren (30) bzw. (33) und (32) bzw. (34) mit $m=2$, $h=0,1$ durchgeführt. Bei der Anfangsiteration und bei der fortlaufenden Rechnung waren zwei bis vier Iterationen erforderlich. (Die Schrittweite h ist hier schon reichlich groß bemessen.)

An den Rändern des Gebietes wurden folgende Formeln benutzt:

1. Oberhalb $y = x$:

$$z^{[1]}_{r,8} = z_{r,7} - \frac{1}{120} (q_{r-1,6} - 8\, q_{r,7} - 5\, q_{r+1,8} + 0{,}05 f^{[0]}_{r,8} + 0{,}35 f_{r,7} -$$
$$- 0{,}1 f_{r,6} + 0{,}35 f_{r+1,8} + 0{,}05 f_{r+1,7} - 0{,}1 f_{r+2,8})$$

für $r = 1, 2, \ldots, 5$,

$$z^{[1]}_{0,8} = z_{0,7} - \frac{1}{120} (6 q_{0,7} + 6 q_{1,8} + 0{,}05 f^{[0]}_{0,8} + 0{,}35 f_{0,7} - 0{,}1 f_{0,6} +$$
$$+ 0{,}35 f_{1,8} + 0{,}05 f_{1,7} - 0{,}1 f_{2,8}).$$

2. Unterhalb $y = x$:

$$z^{[1]}_{r+1,0} = z_{r+1,1} - \frac{1}{120} (5 q_{r,0} + 8 q_{r+1,1} - q_{r+2,2} + 0{,}05 f^{[0]}_{r+1,0} + 0{,}35 f_{r+1,1} -$$
$$- 0{,}1 f_{r+1,2} + 0{,}35 f_{r,0} + 0{,}05 f_{r,1} - 0{,}1 f_{r-1,0})$$

für $r = 2, 3, \ldots, 6$,

$$z^{[1]}_{8,0} = z_{8,1} - \frac{1}{120} (6 q_{2,1} + 6 q_{7,0} + 0{,}05 f^{[0]}_{8,0} + 0{,}35 f_{8,1} - 0{,}1 f_{8,2} +$$
$$+ 0{,}35 f_{7,0} + 0{,}05 f_{7,1} - 0{,}1 f_{6,0}).$$

Tabelle 6. *Näherungen* $z_{\mu,\nu}$

ν	μ = 0	1	2	3	4	5	6	7	8	ν
8	0,696722	0,762689	0,851306	0,969451	1,126103	1,334623	1,613814	1,990515	2,505803	8
7	0,765136	0,828913	0,916031	1,032827	1,187827	1,393816	1,668837	2,037889	2,539741	7
6	0,825539	0,885430	0,968739	1,081362	1,231253	1,430370	1,695606	2,050119	2,529546	6
5	0,877738	0,932034	1,009576	1,115717	1,257976	1,446889	1,698072	2,032708	2,483048	5
4	0,921144	0,968388	1,038502	1,136256	1,268419	1,444427	1,678296	1,989231	2,405675	4
3	0,955381	0,994370	1,055841	1,143745	1,264059	1,425157	1,639394	1,923732	2,303237	3
2	0,980068	1,009901	1,061693	1,138654	1,245899	1,390752	1,583865	1,840016	2,181003	2
1	0,994967	1,015105	1,056255	1,121861	1,215298	1,343072	1,514309	1,741644	2,043774	1
0	1,000000	1,009961	1,040527	1,094168	1,173505	1,283982	1,433290	1,632063	1,896136	0

Fehler $\varepsilon_{\mu,\nu} \cdot 10^6$

ν	μ = 0	1	2	3	4	5	6	7	8	ν
8	+15	+370	+347	+354	+173	+53	−14	−429	0	8
7	+294	+369	+349	+398	+251	+183	+355	0	+376	7
6	+203	+251	+199	+202	−3	−145	0	−281	+13	6
5	+155	+185	+116	+91	+111	0	+132	−95	+176	5
4	+83	+103	+9	−37	0	−84	+12	−45	+143	4
3	+45	+45	+31	0	+25	−38	+30	−79	+17	3
2	+1	−13	0	−21	−10	−29	+4	−167	−169	2
1	−37	0	−277	−3	−3	−43	−49	−282	−396	1
0	0	−89	−284	−6	−6	−43	−39	−253	−345	0

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Institut für Mathematik und Mechanik
der Bergakademie Clausthal

(Eingegangen am 30. Oktober 1959)

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Druck der Universitätsdruckerei H. Stürtz AG., Würzburg
Printed in Germany